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ESSENTIALS

OF

DESCRIPTIVE GEOMETRY

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ESSENTIALS

OF

DESCRIPTIVE GEOMETRY

By

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PREFACE

The aim of this book is to present the essentials of Descriptive Geometry in a clear and concise manner. There has been no attempt to make it an exhaustive treatise on the subject.

Since frequent practice in the applications of the fundamental principles of the subject has been found to produce the best results, numerous original problems are given. Problems involving the application of a particular principle are given at the close of the discussion of that principle, while those of a more general character and requiring more time for their solution are given at the end of the chapter.

By considering tangent lines and tangent planes as the limiting positions of secant lines and secant planes respectively, the troublesome terms "consecutive points" and "consecutive elements" are avoided Tangent planes are discussed in connection with the representation of surfaces. It is believed that this aids the student in securing an early understanding of the fundamental difference between any two classes of surfaces.

Plane sections, intersections and developments of prisms and pyramids are discussed in connection with the same operations applied to cylinders and cones.

While the arrangement and treatment of the subject is the outgrowth of an extended class-room experience, the preliminary notes were developed from year to year by frequent reference to the works of Church, Watson, De la Gournerie, Javary, Pillet, "F. J." and others.

The University of Wisconsin,
Madison, Wis.

January, 1909.

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DESCRIPTIVE GEOMETRY

CHAPTER I

FIRST PRINCIPLES

1. Projection. If from a point S (Fig. 1) straight lines are drawn through a series of points A, B, C . . . , the points a, b, c . . . in which these lines pierce the plane T are projections of the points A, B, C. . . .

S is called the **point of sight.**

A, B, C . . . are points of a magnitude or object in space.

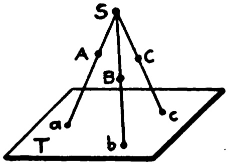
Sa, Sb, Sc . . . . . . are **projecting lines** of the points A, B, C . . . . . .

T is the **plane of projection.**

The projections a, b, c . . . present the same appearance to the eye, situated at the point of sight, as the points A, B, C . . . in space.

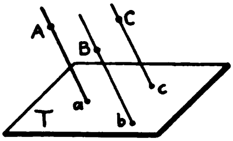
In **perspective,** Fig. 1, the point of sight is at a **finite distance** from the plane of projection. The projecting lines **diverge.**

Fig. 1



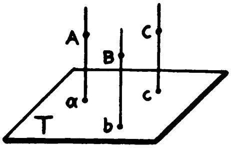
Perspective

Fig. 2



Oblique projection

Fig. 3



Orthographic projection

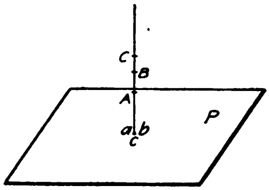
In **oblique** and **orthographic** projection, Figs. 2 and 3, the point of sight is at an **infinite distance** from the plane of projection. The projecting lines are **parallel.**

The projection is **oblique** when the projecting lines are parallel to each other and oblique to the plane of projection.

The projection is **orthographic** when the projecting lines are perpendicular to the plane of projection.

ORTHOGRAPHIC PROJECTION

Fig. 4



a, b, c, projections of points A, B, C, on plane P

2. A point in space is not completely determined by its orthographic projection on one plane, for the distance of the point from the plane is not shown by its projection. All points A, B, C, Fig. 4, which lie in a vertical straight line, have the same projection on a horizontal plane. There are two methods of representing definitely a point in space. One method is to give its projection on a plane and also its distance from that plane. The other method is to use two different planes of projection. The latter is the method commonly used.

3. Planes of projection. In orthographic projection two planes are generally used, Fig. 5, one horizontal and the other vertical, called respectively:

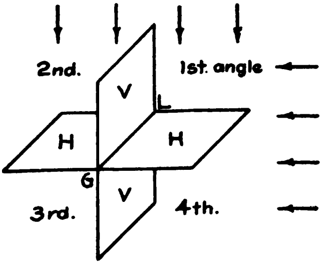
(a) **The horizontal plane of projection** or H.

(b) **The vertical plane of projection** or V.

Their intersection is called the **ground line** or G. L.

Point of sight. The point of sight for the horizontal plane is an infinite distance above H and for the vertical plane it is an infinite distance in front of V. In the horizontal view, the ground line represents the vertical plane seen edgewise, and in the vertical view, the ground line represents the horizontal plane seen edgewise.

Fig. 5



Principal planes of projection

4. The four angles. The right dihedral angles formed by the intersection of the horizontal and vertical planes of projection are known as the **first, second, third** and **fourth** angles, Fig. 5.

The **first angle** is above H and in front of V.

The **second angle** is above H and back of V.

The **third angle** is below H and back of V.

The **fourth angle** is below H and in front of V.

5. The drawing. In order to represent both the horizontal and vertical projection of an object on the same sheet of paper, the planes of projection must be brought together. This is accomplished by keeping one of the planes fixed and revolving the other about the ground line as an axis until the two planes coincide. If the horizontal plane is kept fixed, the upper part of the vertical plane is revolved backward until it coincides with the back part of the horizontal plane. This will be found convenient if the work is done on a drafting board which is in a horizontal position. If the vertical plane is kept fixed, the front part of the horizontal plane is revolved downward until it coincides with the lower part of the vertical plane. This will be found convenient when working on the blackboard. By either method, the first and third angles are opened and the second and fourth closed.

It must be remembered that both the horizontal and vertical projections are made while the planes of projection are still at right angles, the planes being brought together simply for convenience in making both projections on one sheet of paper.

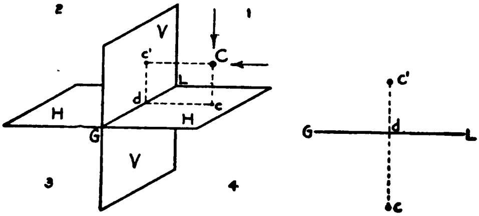
POINTS

6. In Fig. 6, let C be any point in space and H and V the planes of projection. The line Cc, through the point and perpendicular to the horizontal plane is the **horizontal projecting line** of the point and its intersection c, with the horizontal plane, is the **horizontal projection** of the point. Similarly the line Cc′, through the point and perpendicular to the vertical plane, is the **vertical projecting line** of the point and its intersection c′, with the vertical plane, is the **vertical projection** of the point.

The two projections of a point determine its position in space, for their projecting lines intersect in the only point which can have the given projections.

Since the vertical projecting line is parallel to the horizontal plane, then Cc = c′d. That is, **the distance of a point in space from the horizontal plane is equal to the distance of its vertical projection from the ground line.** Also Cc, the horizontal projecting line, is parallel to the vertical plane, making Cc′ = cd. That is, **the distance of a point in space from the vertical plane is equal to the distance of its horizontal projection from the ground line.**

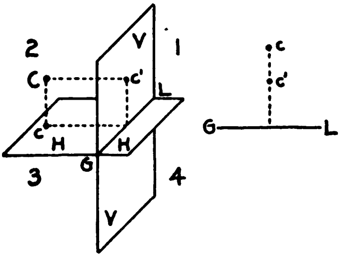
Fig. 6 *Fig. 7*



Point C in 1st angle

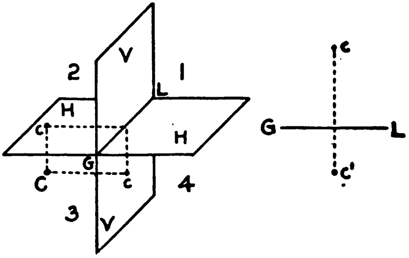
Since c′d remains perpendicular to the ground line during the revolution of the vertical plane from the vertical to the horizontal position, then, on the drawing, Fig. 7, c′c is perpendicular to the ground line, that is, **the two projections of a point in space are in the same line perpendicular to the ground line.**

Fig. 8



Point C in 2nd angle

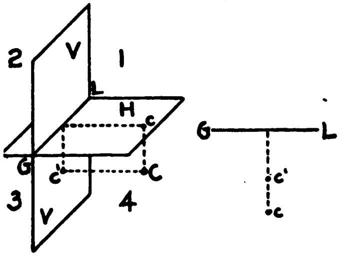
Fig. 9



Point C in 3rd angle

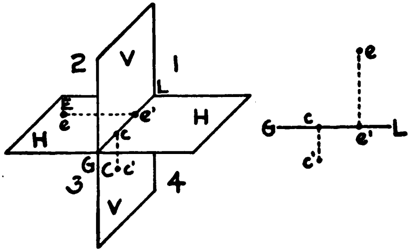
7. In Figs. 8, 9, 10 and 11, the planes of projection are shown and a point in different positions with reference to these planes. At the right of each figure is shown the orthographic projections of the point after the planes have been brought together. In the solution of problems, the projections are to be shown only in their final positions. The sketch to the left of each of these figures is used only to give a better idea of the position of the point with reference to the planes of projection. Students should avoid these sketches as much as possible and should endeavor to picture the points in space from the projections after the planes are brought into coincidence.

Fig. 10



Point C in 4th angle

Fig. 11



Point E in H back of V
Point C in V below H

It does not matter in which angle the object is placed, the point of sight for the horizontal view is always above H, and for the vertical view it is always in front of V.

8. Notation. Points in space are designated by capital letters as A, B, C, etc.

Horizontal projections of points, small letters as a, b, c, etc.

Vertical projections of points, small letters with prime marks as a′, b′, c′, etc.

Ground line, full.

The line joining the projections of a point will be drawn fine, full in pencil, and fine, full, red in ink.

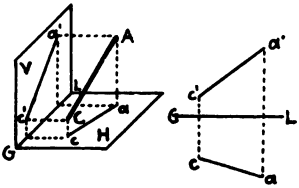
9. Problems. Draw the projections of a point in each of the following positions, and then from its projections, picture the point in space. If working at the blackboard, picture from V; if on a drafting board, picture from H.

1. In the first angle.
2. In the upper part of V.
3. In the second angle equidistant from H and V.
4. In the back part of H.
5. In the third angle.
6. In the front part of H.
7. In the fourth angle.
8. In the ground line.

LINES

10. In Fig. 12, let AC be any straight line in space. The plane ACc, through the line and perpendicular to the horizontal plane, is the **horizontal projecting plane** of the line. Its intersection ac with the horizontal plane, is the **horizontal projection** of the line. Similarly the plane ACc′, through the line and perpendicular to the vertical plane, is the **vertical projecting plane** of the line. Its intersection a′ c′ with the vertical plane is the **vertical projection** of the line.

Fig. 12



Projections of a line

The two projections of a straight line determine its position in space, for their projecting planes intersect in the only line which can have the given projections.

When the two projections of a line are in the same perpendicular to the ground line, the projecting planes will coincide and the line is undetermined.

If, however, the projections of any two points of a straight line are given, the line is always determined.

At the right of Fig. 12, is the orthographic projection of the line after the planes of projection have been brought into coincidence.

11. Projections of straight lines in various positions with reference to the planes of projection.

If the line is oblique to the planes of projection, the projections of the line will be inclined to the ground line, Fig. 12.

If a line is parallel to the ground line, its projections will be parallel to the ground line.

If a line is perpendicular to the horizontal plane, its horizontal projection is a point and its vertical projection is a straight line perpendicular to the ground line.

If a line is perpendicular to the vertical plane, its vertical projection is a point and its horizontal projection is a straight line perpendicular to the ground line.

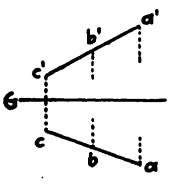
If a line is parallel to the horizontal plane and oblique to the vertical plane, its vertical projection will be parallel to the ground line and its horizontal projection will be inclined to the ground line.

If a line is parallel to the vertical plane and oblique to the horizontal plane, its horizontal projection will be parallel to the ground line and its vertical projection will be inclined to the ground line.

12. Point on line. If a point is on a line in space, the horizontal projection of the point will be on the horizontal projection of the line and the vertical projection of the point will be on the vertical projection of the line, Fig. 13.

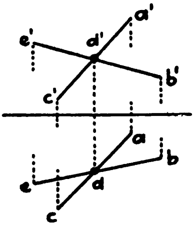
Intersecting lines. If two lines intersect at a point in space,

Fig. 13



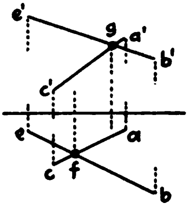
Point B on line AC

Fig. 14



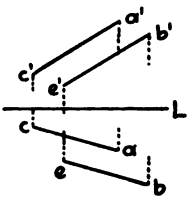
Lines Intersecting at D

Fig. 15



Lines not Intersecting

Fig. 16



Parallel lines

the horizontal projections of the lines will intersect in the horizontal projections of the point and the vertical projections of the lines will intersect in the vertical projection of the point, Fig. 14. The projections of the common point D must lie in the same perpendicular to the ground line.

If two lines do not intersect in space, their projections will not intersect in the same perpendicular to the ground line, Fig. 15.

Parallel lines. If two lines are parallel in space, their projections on the same plane are parallel, Fig. 16.

13. Notation. Lines in space are designated by capital letters, as AB, CD, MN, etc.

Horizontal projections of lines, small letters as ab, cd, mn, etc. Vertical projections of lines, small letters with prime marks, as a′b′, c′d′, m′n′, etc.

The planes of projection will be considered transparent.

14. Problems. Draw the projections of a line in each of the following positions and then from its projections, picture the line in space. On the blackboard, picture from V; on the drafting board, picture from H.

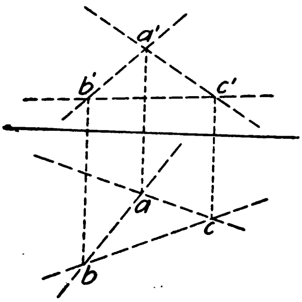
1. In the first angle, parallel to H and V.
2. In the second angle, parallel to V and oblique to H.
3. In the back part of H and inclined to V.
4. In the third angle perpendicular to H.
5. In the third angle, oblique to H and V. Draw the projections of a point on this line.
6. In the first angle, oblique to H and V, and in a plane perpendicular to G. L.
7. In a plane bisecting the fourth angle.
8. Two intersecting lines in the third angle.
9. Two parallel lines, one in the first and the other in the second angle.

PLANES

15. In space, a plane is fixed by three points not in the same straight line, a point and a line, two intersecting lines or two parallel lines. If the horizontal and vertical projections of these magnitudes are given, the plane which they determine in space will be definitely located.

The points A, B and C, Fig. 17, fix a plane in space. It is evident that a straight line joining A and B, B and C or A and C will lie in this plane. If any of these lines pierce the horizontal plane of projection, the piercing point must be on the line of intersection of the plane of the given points A, B and C with the horizontal plane. Likewise, if the lines pierce the vertical plane of projection, the piercing points must be on the line of intersection of the plane of the given points with the vertical plane.

Fig. 17



Points A, B, C, fix a plane

If a point and line are given by their projections, they will fix a plane. Other straight lines of the plane can be found by joining the given point with points on the given line.

If the plane is fixed by two straight lines, either parallel or intersecting, any number of other straight lines of the plane can be found by joining a point in one of the given lines with a point in the other.

16. Traces of planes. In Fig. 18, let tTt′ be an oblique plane. The intersection tT with the horizontal plane is called **the horizontal trace of the plane;** and the intersection t′T with the vertical plane is called the **vertical trace of the plane.**

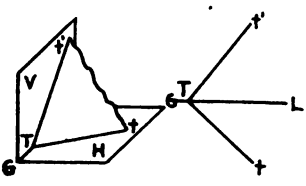


Fig. 18

The traces of a plane, being two intersecting or parallel lines, determine its position in space.

If a plane intersects the ground line, its traces will intersect the ground line at the same point.

If a plane is parallel to the ground line and oblique to the planes of projection, its traces will be parallel to the ground line.

If a plane simply contains the ground line, its position is not determined.

If a plane is perpendicular to the horizontal plane, its vertical trace will be perpendicular to the ground line.

Likewise, if a plane is perpendicular to the vertical plane, its horizontal trace will be perpendicular to the ground line.

If a plane is perpendicular to both of the planes of projection and therefore to the ground line, both of its traces will be perpendicular to the ground line.

If a plane is parallel to the horizontal plane, its vertical trace will be parallel to the ground line.

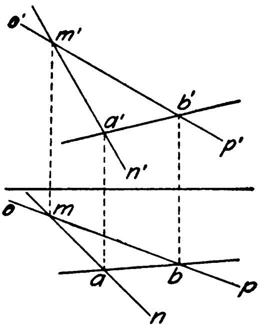
Likewise, if a plane is parallel to the vertical plane, its horizontal trace will be parallel to the ground line.

17. Notation. Planes, given by their traces, are referred to by capital letters, as S, T, etc.

Horizontal traces of planes are designated sS, tT, etc., Fig. 18. The capital letter always being the point where the trace cuts the ground line.

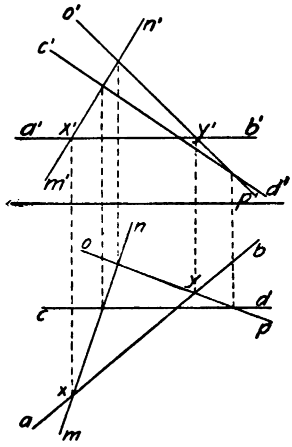
Vertical traces of planes are designated s′S, t′T, etc., Fig. 18.

Fig. 19



AB line in plane of MN and OP

Fig. 20



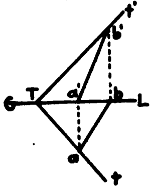
AB in plane and parallel to H
CD in plane and parallel to V

When the traces of a plane do not cut the ground line within the limits of the drawing, the horizontal trace will be designated ss and the vertical trace s′s′.

18. Line in plane. The two intersecting lines MN and OP, Fig. 19, fix a plane. The straight line AB, joining the point A in MN with the point B in OP, is another line of the plane. ab is its horizontal and a′ b′ its vertical projection.

In Fig. 20, AB is a line parallel to H and lying in the plane of the lines MN and OP. Its vertical projection a′ b′ is first drawn and then its horizontal projection ab is found by finding the horizontal projections of the points X and Y where it crosses the given lines MN and OP. CD is a line parallel to V and lying in the same plane. Its horizontal projection cd is first drawn and then its vertical projection is found. These lines which lie in a plane and are parallel to H or V play an important part in the solution of many problems.

Fig. 21

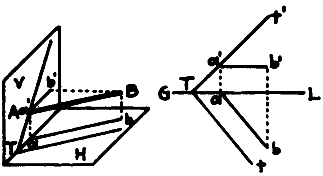


Line AB in plane T

19. If the plane is given by its traces, as in Fig. 21, a straight line joining the point A in the horizontal trace with the point B in the vertical trace will be a line of the plane T. The point A, being on the horizontal trace, has a and a′ for projections and B, being on the vertical trace, has b and b′ for projections. Then ab and a′ b′ are projections of the required line.

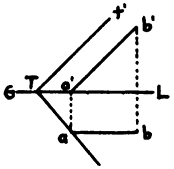
A line AB, Fig. 22, which lies in a given plane and is parallel to the horizontal will be parallel to the horizontal trace of the plane. The vertical projection of this line will be parallel to the ground line and the horizontal

Fig. 22



Line AB parallel to H

Fig. 23



Line AB parallel to V

projection will be parallel to the horizontal trace of the plane. The point A is where the line cuts the vertical trace of the plane.

A line AB, Fig. 23, which lies in the given plane and is parallel to the vertical will be parallel to the vertical trace of the plane. The horizontal projection of this line will be parallel to the ground line and the vertical projection will be parallel to the vertical trace of the plane. The point A is where the line cuts the horizontal trace.

20. If one projection of a line which lies in a plane is given, the other projection can be found. To do this find the other projections of the points where this line intersects the lines which fix the plane and join them by a straight line.

If one projection of a point which lies in a plane is given, the other projection is found by finding the projections of a line of the plane passing through the point and then locating the other projection of the point on the other projection of the line.

21. Problems.

1. Having given the horizontal projection of a line which lies in a plane fixed by two intersecting lines, to find the vertical projection.
2. Having given the vertical projection of a line which lies in a plane fixed by its traces, to find the horizontal projection.
3. Having given one projection of a point which lies in a given plane, to find the other projection.
4. Find the projections of a point which lies in a given plane and is a given distance from V.
5. Find the projections of a point which lies in a given plane and is a given distance from H.
6. Find the projections of a point which lies in a given plane and is a given distance from H and V.

SUPPLEMENTARY PLANES OF PROJECTION

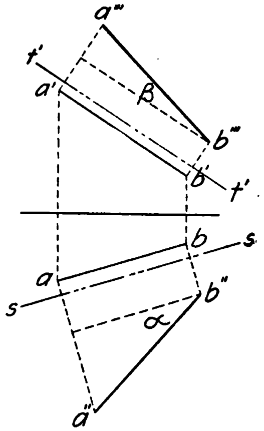
22. Thus far only the horizontal and vertical planes of projection have been used. It will, however, frequently be found convenient to make use of other planes which are in a simpler position with reference to the object than either H or V. These are called supplementary planes of projection.

The positions of these supplementary planes depend upon the position and form of the object. They are usually perpendicular to H but they may be taken perpendicular to V or in any position in which a simple view of the object can be obtained. A plane which is perpendicular to the ground line is called a **profile plane.**

The use of such a plane is shown in Fig 24 where AB is a given oblique line. The horizontal and vertical projections of the line, ab and $a'b'$, do not show its true length or the true angles which it makes with the planes of projection. $a''b''$ is the projection of this line on a plane S which is parallel to the line and perpendicular to H. Since the line is parallel to the plane S, its projection a'' b'' on this plane is the true length of the line and the angle α is the true angle which the line makes with the horizontal plane.

$a'''b'''$ is the projection of the line on a plane T which is parallel to the line and perpendicular to V. This projection shows the true length of the line and the angle β which the line makes with the vertical plane of projection.

Fig. 24



True length of line and angles it makes with H and V

Since a profile plane is perpendicular to the ground line, it can be revolved into either H or V after the projection of the object is made upon it. It is convenient to use this plane when an object such as a prism or cylinder has its axis parallel to the ground line.

REVOLUTION AND COUNTER REVOLUTION OF OBJECTS

23. An object is said to revolve about a straight line as an axis when each of its points moves in the circumference of a circle whose center is in the axis and whose plane is perpendicular to the axis.

When an object is revolved about a straight line as an axis, the relative position of its points is not changed. The object can thus be brought into a simpler position with reference to the planes of projection. The projections of the object in this position are easily found and from these projections the projections of the object in its original position are found by the counter revolution of its points.

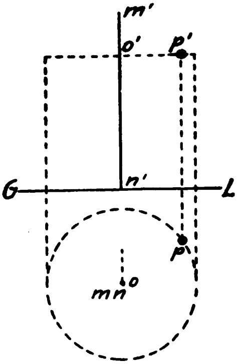
REVOLUTION OF A POINT ABOUT AN AXIS

24. When the axis is perpendicular to H.

In Fig. 25, let MN be the axis and P the given point.

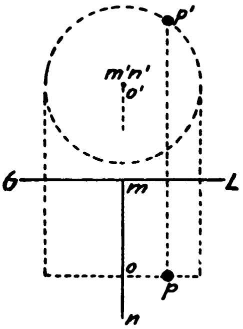
The point P will move in the circumference of a circle whose center is at O and whose radius is OP (Art. 23). The plane of the path of this point is perpendicular to MN and therefore parallel to H. Therefore the horizontal projection of the path is a circle with a radius op and the vertical projection is a straight line through p′ parallel to the ground line.

Fig. 25



Point P revolved about MN as axis

Fig. 26



Point P revolved about MN as axis

25. When the axis is perpendicular to V.

In Fig. 26, let MN be the axis and P the given point.

The point P will move in the circumference of a circle whose center is at O and whose radius is OP (Art. 23). The plane of the path of this point is perpendicular to MN and therefore parallel to V. Therefore the vertical projection of the path is a circle with radius o′ p′ and the horizontal projection is a straight line through p parallel to the ground line.

26. To revolve a point in space into H about an axis which lies in H and is oblique to V.

In Fig. 27, let MN be the axis and P the given point.

The point P will move in the circumference of a circle whose center is at O, and whose radius is OP (Art. 23). The plane of the path of this point is perpendicular to MN, and therefore perpendicular to H, cutting H in the line op. The point P will fall in the horizontal plane either at P_1 or P_2 on the trace op, at a distance from o equal to OP, the radius of the circle.

Fig. 28 shows the orthographic projection of this construction.

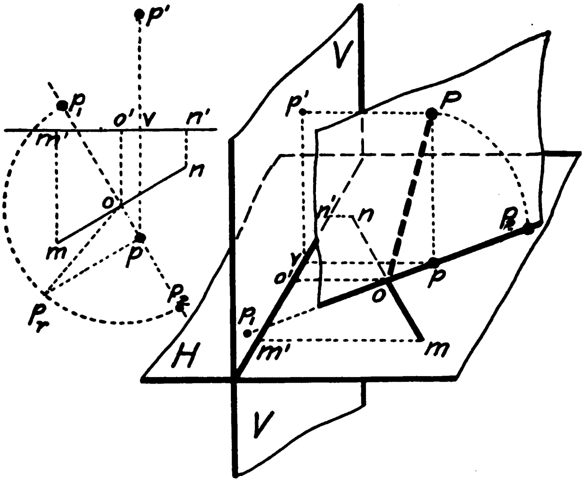


Fig. 28 *Fig. 27*

Point P revolved into H about MN as axis

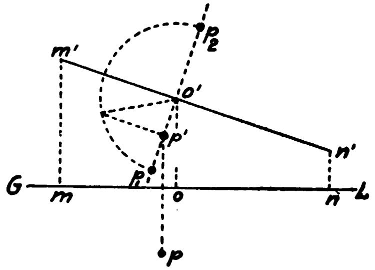
The distance from p′ to the ground line, Fig. 28, is equal to the distance Pp, Fig. 27. Then oP_1 and oP_2, Fig 28, are each equal to the hypothenuse of a right-triangle of which op is the base and from p′ to the ground line is the altitude. In Fig. 28, the plane of the circle is imagined turned to the left into H. Then $pp_r = p'v$ and op_r is the true length of the radius.

If the point is directly above the axis, it is evident that the base of the triangle is zero, and the radius of the circle then becomes equal to the altitude which is the distance of the point above the horizontal plane.

27. To revolve a point in space into V about an axis which lies in V and is oblique to H.

In Fig. 29, let MN be the axis and P the given point.

Fig. 29



Point P revolved into V about MN as axis

The point P will move in the circumference of a circle whose center is at O and whose radius is OP (Art. 23). The plane of the path of this point is perpendicular to MN, and therefore pependicular to V, cutting V in the line o′ p′. The point P will fall in the vertical plane either at P_1 or P_2 on the trace o′ p′, at a distance from o′ equal to the hypothenuse of a right-triangle of which o′ p′ is the base and from p to the ground line is the altitude.

In this case the student will find it convenient to hold the page in a vertical position and picture the magnitudes in front of their vertical projections.

If the point is directly in front of the axis, it is evident that the base of the triangle is zero and the radius of the circle then becomes equal to the altitude which is the distance of the point from the vertical plane.

28. Problems.

1. Revolve a point in the first angle through an angle of 45

degrees about an axis perpendicular to H and show its projections in the new position.

2. Revolve a point in the fourth angle into H about an axis in H.

3. Revolve a point in the second angle into V about an axis in V.

4. Revolve a point in the third angle into H about the ground line as an axis.

5. Having a given point and an axis in H, revolve the point about the axis until it is 1'' above H and show its projections in this position.

6. Having given a point in V, revolve the point about the ground line as an axis until it is 2'' from V, and show its projections in this position.

LINE NOTATION—PENCIL

29. The planes of projection are considered transparent.

Hidden traces and traces of auxiliary planes, dot-dash, thus:

——— – – ——— – – ——— – – ———

Retrace required lines so that they stand out from the construction making hidden lines dashed. (Dashes about $\frac{1}{8}''$ long and $\frac{1}{16}''$ apart.)

All other construction is to be in very fine, full lines.

LINE NOTATION—INK

30. Traces of given planes, when visible, fine, full, black lines; when invisible, fine, dot-dash, black lines.

Traces of required planes, when visible, heavy, full, black lines; when invisible, heavy, dot-dash, black lines.

Traces of auxiliary planes, fine, dot-dash, red lines.

Given lines, when visible, fine, full, black lines; when invisible, fine, dashed, black lines.

Required lines when visible, heavy, full, black lines; when invisible, heavy, dashed, black lines.

All other construction is to be in fine, full, red lines.

CHAPTER II

ELEMENTARY PROBLEMS RELATING TO THE POINT, STRAIGHT LINE AND PLANE

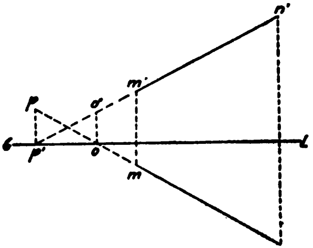
31. To find where a line which is given by its projections pierces the planes of projection.

Let MN, Fig. 30, be the line given by its projections mn and m′n′.

First. To find where the line pierces V.

Analysis. In Fig. 30, let the paper be the horizontal plane and let the vertical plane stand perpendicular to the paper on G. L. As a point moves along the line from N toward M its horizontal projection follows the line nm, approaching the ground line. The distance of the horizontal projection of the point from the ground line is always the distance from the point in space to the vertical plane. Therefore when the horizontal projection of the point is at o, on G. L., the point in space is on the vertical plane. This is the point where the line MN pierces V.

Fig. 30



Line MN pierces V at O and H at P

Construction. Extend mn to o and construct oo′ perpendicular to the ground line. The point o′, where this perpendicular intersects m′n′ extended, is the required point.

Second. To find where the line pierces H.

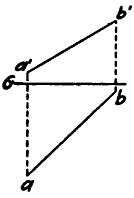
Analysis. Imagine the planes of projection as before. As a point moves along the line from N toward M its vertical projection follows the line n′m′, approaching the ground line. The distance of the vertical projection of the point from the ground line is always the distance from the point in space to the horizontal plane. Therefore when the vertical projection of the point is at p′, on G. L., the point in space is on the horizontal plane. This is the point where the line MN pierces H.

Construction. Extend m′n′ to p′ and construct p′p perpendicular to the ground line. The point p, where this perpendicular intersects mn extended, is the required point.

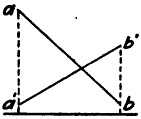
32. Problems.

1. Let the problem be solved when the line is in the third angle.

2, 3, 4, 5. Find where the lines whose projections are given in the following figures pierce the horizontal and vertical planes of projection.



2



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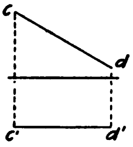

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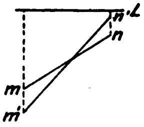


5

6. Having given the horizontal and vertical piercing points of a line, to draw the projections of the line.

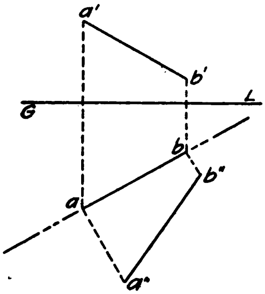
33. To find the length of a straight line.

Let AB, Fig. 31, be the given line.

Analysis. First Method. Revolve the horizontal projecting plane of the line about its horizontal trace until it coincides with H. The given line in its revolved position will be shown in its true length.

Construction. Revolve the line about ab, the horizontal trace of its horizontal projecting plane. The points A and B will fall at a″ and b″ respectively, at distances from a and b equal to the height of the points A and B from H (Art. 26); a″b″ is the required length, since the points do not change their relative position during the revolution (Art. 23).

Fig. 31



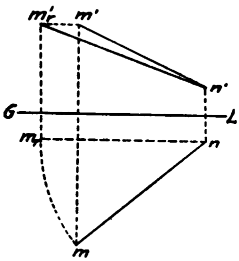
a″b″ true length of line

The problem can also be solved by revolving the vertical projecting plane of the line about its vertical trace until it coincides with V.

34. Second method for the same problem.

Let MN, Fig. 32, be the given line.

Fig. 32



$n'm_r'$ true length of line

Analysis. Revolve the horizontal projecting plane of the line about the horizontal projecting line of one of its points until it is parallel to V. In this position the line will be projected on V in its true length.

Construction. Revolve the horizontal projecting plane of MN about the horizontal projecting line of N as an axis. The point N remains fixed, being in the axis, and M describes a horizontal circle with radius nm (Art. 24). The vertical projection of this circle is a straight line through m′ parallel to the ground line. The line will be parallel to V when m reaches m_r, nm_r being parallel to the ground line. The vertical view $n'm'_r$ in this position is the true length of the line.

The line can also be revolved until it is parallel to H about the vertical projecting line of one of its points. In this position the horizontal view will be the true length of the line.

35. Problems.

1. Find the length of a line which is in the third angle.
2. Find the length of a straight line joining a point in the first angle with one in the second angle.
3. Find the length of a straight line joining a point in the first angle with one in the third angle.
4. Find the length of a straight line joining any two points in a profile plane.
5. Find the length of a straight line joining a point in the second angle with one in the fourth angle.
6. Find the length of a straight line joining a point in G. L. with a point in the second angle.

7. From a point on a line, given by its projections, lay off along the line a length of two units.

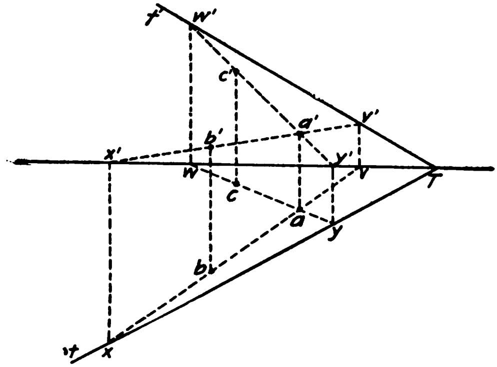
Find where the lines in the above problems pierce H and V.

36. To find the traces of a plane containing three given points.

Let A, B and C, Fig. 33, be the given points.

Analysis. A straight line joining two of the given points will lie in the required plane. One of these points joined with the third point will give another line of the plane. These two lines pierce H in two points which fix the horizontal trace of the plane. These lines also pierce V in two points which determine the vertical trace of the plane. Therefore the traces of the plane are both determined.

Fig. 33



Plane T contains points A, B and C

Construction. ab and a′b′, ac and a′c′ are the horizontal and vertical projections of the lines joining A with B and A with C. AB pierces H at x, and AC pierces H at y (Art. 31). Therefore xy is the horizontal trace of the required plane. Also AB pierces V at v′ and AC pierces V at w′. Hence w′v′ is the required vertical trace.

xy and v′w′ must intersect at the ground line.

If two lines lie in a plane, a line connecting any point of one of these lines with any point of the other will also lie in that plane.

37. Problems.

1. Find the traces of a plane containing three given points in the third angle.

2. Find the traces of a plane containing two intersecting

lines; one oblique to H and V and one parallel to the ground line.

3. Find the traces of a plane which will contain two given parallel lines.

4. Pass a plane through three points when one of them is on the ground line.

5. Pass a plane through a line which is oblique to H and V and a point which lies in V.

6. Pass a plane through two intersecting lines; one is parallel to H and oblique to V, and one is parallel to V and oblique to H.

7. Pass a plane through two lines which do not cut H or V within the limits of the drawing.

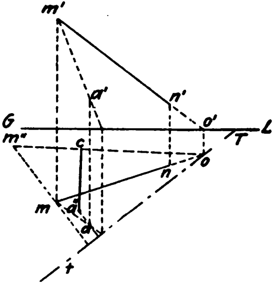
8. Find the traces of a plane containing three given points; one in H, one in V and one in the first angle.

9. Pass a plane through two lines which intersect on the ground line.

38. To find the distance from a given point to a given straight line.

In Fig. 34, let A be the given point and MN the given line.

Fig. 34



a"c distance from A to line MN

Analysis. Find the traces of a plane containing the given point and line (Art. 36). Revolve the point and line into H about the horizontal trace or into V about the vertical trace of this plane (Art. 26). Since the point and line do not change their relative position during the revolution (Art. 23), the distance between them will be shown in its true length on the horizontal or on the vertical plane of projection.

Construction. Join A with some point of MN, as M, and find the traces of a plane containing the lines MN and MA. tT is the horizontal trace of this plane. When the plane is revolved into H about tT as an axis, A falls at a'', M at m'' and o remains fixed, being on the axis (Art. 26). Then $a''c$, perpendicular to $m''o$, is the true distance from the point A to the line MN.

39. Problems.

1. Let the problem be solved when both the point and the line are in the third angle.

2. Find the distance from a point on V to a line in the first angle.

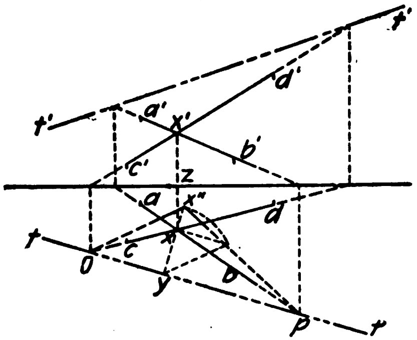
3. Draw the projections of a line which passes through a given point and makes 30° with a given line.

4. Through a point on a given line draw a line which makes α° with the given line.

40. To find the angle between two intersecting straight lines.

Let AB and CD, Fig. 35, be the given lines.

Fig. 35



ox″p true angle between AB and CD

Analysis. Since the lines intersect, the traces of a plane containing them can be found. If this plane be revolved into H about its horizontal trace, and the revolved positions of the lines be found, the angle will be shown in its true size, since the lines do not change their relative positions during the revolution (Art. 23).

Construction. tt is the horizontal and t′t′ the vertical trace of the plane containing the lines AB and CD (Art. 36). Revolve the plane T into H about tt as an axis (Art. 26). x moves to x″ and o and p remain fixed, being in the axis. yx″ is equal in length to the hypothenuse of a right-triangle having xy as the base and zx′ as the altitude. Then ox″ is the revolved position of CD and px″ is the revolved position of AB. Hence ox″p is the required angle.

The true size of the angle can be found by revolving the plane T into V about t′t′ as an axis. The axis might also be taken in the plane T parallel with either t t or t′t′ and the true size of the angle found by revolving the plane until it is parallel with one of the planes of projection.

The projections of the angle may be less, equal to or greater than the angle itself.

To bisect an angle. When the angle is shown in its true size, the bisector can be drawn. The bisector can then be revolved back to the original position with the plane T and its projections found.

41. Problems.

1. In Art. 40, find the angle by revolving into V about t′t′ as an axis.

2. Bisect the angle between two intersecting lines in the third angle.

3. Draw the projections of a pyramid with base on H and find the angle between any two lateral edges, also the angle between a lateral edge and a side of the base.

4. Find the angle between two intersecting lines when one of them is parallel to H.

5. Find the angle between two lines when one is perpendicular to V.

6. Find the angle between two intersecting lines when one is oblique to H and V and the other parallel to G. L.

7. Find the angle between the traces of a plane.

8. Two lines intersect on the ground line; find the angle between them.

9. Find the angle between two intersecting lines when one is oblique to H and V and one lies in a profile plane.

10. Draw the projections of a triangle and bisect the angles.

11. Find the angle between two lines by revolving parallel to H about an axis parallel to H in the plane of the angle.

12. Find the angle by revolving parallel to V about an axis parallel to V in the plane of the angle.

13. Find the angle between the G. L. and a line which cuts it.

42. Through a given point to pass a plane which will be parallel to two given straight lines.

Analysis. Through the given point drawn a line parallel to each of the given lines. A plane containing these two lines is the required plane.

Let the construction be made according to the given analysis.

43. Problems.

1. Through a given point, pass a plane parallel to two lines, one of which is parallel to G. L.

2. Through a given line, pass a plane parallel to another given line.

3. Through a given point, pass a plane parallel to a given plane.

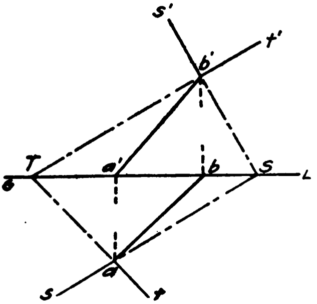
4. Through a given line, pass a plane parallel to G. L.

44. To find the line of intersection of two planes which are given by their traces.

Let T and S, Fig. 36, be the given planes.

Analysis. The intersection is a straight line lying in each plane. If any two of its points be located and joined by a straight line, it will be the required intersection. One point of the line is at the intersection of the horizontal traces, and another is at the intersection of the vertical traces. Joining these two points will give the required intersection.

Fig. 36



AB line of intersection of planes T and S

Construction. a and a′ are the projections of the intersection of the horizontal traces, and b and b′ are the projections of the intersection of the vertical traces. Then ab is the horizontal and a′b′ the vertical projection of the required line of intersection of the planes T and S.

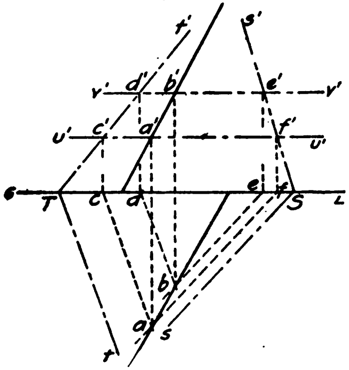
45. The same problem as Art. 44 when the traces do not intersect within the limits of the drawing.

Let T and S, Fig. 37, be the given planes.

Analysis. If a plane be passed parallel to H it will cut lines from T and S which will be parallel to their horizontal traces.

These lines will intersect in a point of the required intersection. By passing another plane parallel to H, another point of the intersection is found. The line joining these two points is the required intersection.

Fig. 37



AB line of intersection of planes T and S

The problem can be solved by passing the auxiliary planes parallel to V.

Construction. Let U and V be the two auxiliary planes taken parallel to H. ca and c'a' are the two projections of the line cut from T by the plane U, and fa and f'a' are the projections of the line cut from S. a is the horizontal projection and a' the vertical projection of their intersection. In a similar way the points b and b' are found. Then ab is the horizontal and a'b' is the vertical projection of the required intersection.

46. Problems.

1. Find the line of intersection of two planes in the third angle.

2. Find the intersection of two planes when one is oblique to H and V and the other perpendicular to H and oblique to V.

3. Find the intersection of two planes when one is oblique to H and V and the other parallel to H.

4. Find the intersection of two planes when one is oblique to H and V and the other parallel to V.

5. Find the intersection of two planes when they are both perpendicular to V or when they are both perpendicular to H.

6. Find the intersection of two planes when one is oblique to the ground line and one parallel to the ground line, crossing the first angle.

7. The same as problem 6, when the plane parallel to the ground line crosses the second angle.

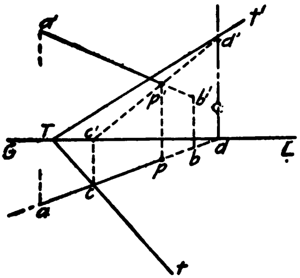
8. The same as problem 6, when the plane parallel to the ground line crosses the fourth angle.

47. To find the point in which a given straight line pierces a given plane.

Case I. When the plane is given by its traces.

Let AB, Fig. 38, be the given line and T the given plane.

Fig. 38



AB pierces T at P

Analysis. If a plane be passed containing the given line it will cut a line from the given plane which will contain the required point. The point must also be in the given line. It must therefore be at the intersection of the given line with the line of intersection of the two planes.

Construction. Let the auxiliary plane be the horizontal projecting plane of the line; ad is the horizontal and dd′ the vertical trace of this plane. cd and c′d′ are the projections of the line of intersection of this plane with T (Art. 44). c′d′ cuts a′b′, the vertical projection of the given line, at p′ which is the vertical projection of the required point. The horizontal projection of the point is at p.

The point can be found by using the vertical projecting plane of the line AB.

48. Problems.

1. Let the problem be solved with the line and plane in the third angle.

2. Find the point where a line which is parallel to G. L. pierces a plane which is oblique to H and V.

3. Find the point where a line which is parallel to H and oblique to V pierces a plane which is parallel to G. L.

4. Find the point where a line which lies in a profile plane pierces a plane which is parallel to G. L.

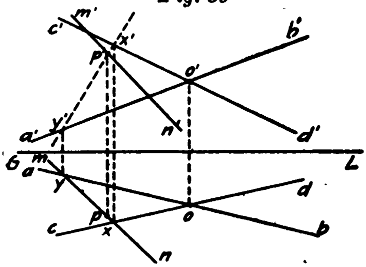
5. Find the length of the part of a line intercepted between two planes.

6. Given three lines, no two of which lie in the same plane; find the projections of a line which touches all three of them.

7. Find the intersection of an oblique triangular pyramid with a given oblique plane.

49. Case II. When the plane is given by any two of its straight lines.

Fig. 39



MN pierces plane of AB and CD at P

Let AB and CD, Fig. 39, be the lines which determine the plane and MN the given line.

Analysis. Find the points where the lines which determine the plane pierce the horizontal or vertical projecting plane of the given line. The line joining these two points will cut the given line in the required point.

Construction. AB pierces the horizontal projecting plane of MN at Y and CD pierces it at X. x'y', the vertical projection of the line joining these two points, intersects m'n' at p', the vertical projection of the required point; p is its horizontal projection.

Remarks. This construction can be used when the plane is given by three points, a point and a line or two parallel lines.

Case I is the same as Case II when the traces are considered as the given lines of the plane.

50. Problems.

1. Let the problem be solved when all of the lines are in the third angle.

2. Find where a line pierces the plane of two parallel lines without finding the traces of the plane.

3. Find where a line pierces the plane of a point and a line without finding the traces of the plane.

4. Find where a line pierces the plane of three points without finding the traces of the plane.

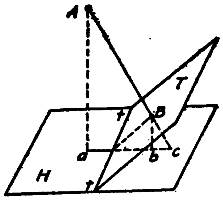
5. Find the line of intersection of two planes each given by two intersecting lines without finding the traces of the planes.

6. Find the intersection of a prism and a pyramid by the above method.

51. If a line is perpendicular to a plane, the projections of the line will be perpendicular to the respective traces of the plane.

Let AB be the line and T the plane, Fig. 40.

Fig. 40



AB perpendicular to plane T

The horizontal projecting plane of AB contains AB and Aa and is perpendicular to both T and H, and therefore to their intersection tt. Hence tt is perpendicular to every line of the horizontal projecting plane, and therefore perpendicular to ab, which is the horizontal projection of AB.

In a similar way it is shown that the vertical projection of the line is perpendicular to the vertical trace of the plane.

Conversely—If the projections of a line are perpendicular to the respective traces of a plane, the line is, in general, perpendicular to the plane.

For if a plane is taken perpendicular to H through ab, it will be perpendicular to tt, and therefore to T. In a similar way it is shown that the plane perpendicular to V, containing the vertical projection of the line, is perpendicular to T. Hence the intersection of the two planes, which is the line AB, is perpendicular to T.

This is not true when the plane is parallel to G. L.

52. To find the distance from a point to a plane.

Analysis. Draw a perpendicular from the given point to the given plane (Art. 51), and find where it pierces the plane (Art. 47). The length of the perpendicular from the given point to the piercing point is the required distance.

Let the construction be made in accordance with the given analysis.

53. Problems.

1. Find the distance from a point to a plane which is parallel to G. L.

2. Find the distance from a point to a plane which is perpendicular to V and oblique to H.

3. Find the distance from a point to a plane which is perpendicular to H and oblique to V.

4. Find the projections of a point which is 4″ from a given plane.

5. Find the distance from a point on the ground line to a given plane.

6. Find the traces of an oblique plane which is parallel to and 4″ from a given oblique plane.

7. Find a point on the ground line which is 4″ from a given oblique plane.

54. To project a given straight line on any oblique plane.

Analysis. Through any two points of the line, erect perpendiculars to the given plane. A line joining the points where these perpendiculars pierce the plane will be the required projection.

Let the construction be made in accordance with the above analysis.

55. Problems.

1. Find the projection of a line which is parallel to H and oblique to V on a plane which is oblique to the ground line.

2. Find the projection of a line which is parallel to V and oblique to H on a plane which is oblique to the ground line.

3. The horizontal and vertical projections of a line are parallel respectively to the traces of a plane. Find the projection of the line on the plane.

4. Find the projection of a line which is parallel to the ground line on a given oblique plane.

5. Find the projection of a line which is oblique to H and V on a plane which is parallel to the ground line.

6. Find the projection of the ground line on a given oblique plane.

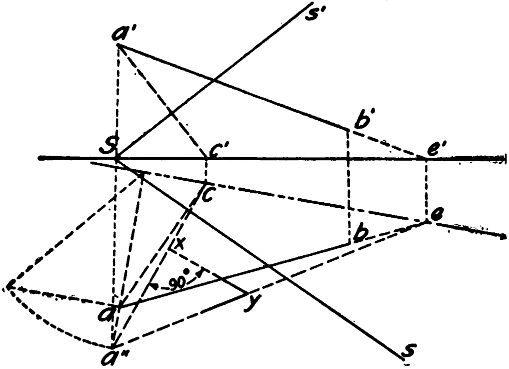
7. Find the projection of a line which lies on V on a plane which is parallel to the ground line.

8. Find the projection of a line which lies in a profile plane on a plane which is oblique to the ground line.

56. To find the angle which a given straight line makes with a given plane.

Let AB, Fig. 41, be the given line and S the given plane.

Fig. 41



xya" angle AB makes with plane S

Analysis. The angle which a line makes with a given plane is understood to be the angle which the line makes with its projection on that plane. If a perpendicular be dropped to the plane from some point in the given line, this perpendicular, the given line and the projection of the line on the plane form a right-triangle. One of the acute angles of this triangle, the one between the given line and its projection on the plane, is the required angle; the other acute angle is the angle between the given line and the perpendicular to the plane and is the complement of the required angle. Therefore if the angle between the given line and a perpendicular to the plane from any point of the line be found and its complement constructed, this will be the required angle.

Construction. Draw a perpendicular to the plane S from the point A; ac and a′c′ are the projections of this perpendicular (Art. 51). ce is the horizontal trace of the plane containing the lines AC and AB (Art. 36). ca″e is the true size of the angle between AC and AB (Art. 40). xya″, the complement of ca″e, is the required angle.

57. Problems.

1. Let the problem be solved when the line and plane are both in the third angle.

2. Find the angle which a line parallel to the ground line makes with a plane which is oblique to the ground line.

3. Find the angle which a line parallel to H and oblique to V makes with a plane which is oblique to the ground line.

4. Find the angles which a given line makes with the planes of projection.

5. Draw the projections of a line which makes 30° with H and is oblique to V.

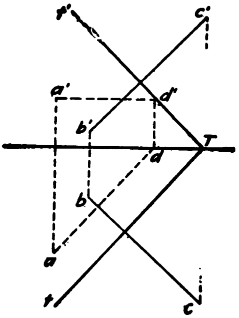
6. Draw the projections of a line which makes 60° with V and is oblique to H.

7. Draw the projections of a line which makes 30° with H and 45° with V.

8. Can a line make 45° with H and 60° with V?

58. Through a given point, to pass a plane perpendicular to a given straight line.

Fig. 42



Plane T perpendicular to line BC

Let A be the given point and BC the given line, Fig. 42.

Analysis. The directions of the traces of the required plane are known; the horizontal trace being perpendicular to the horizontal projection of the line and the vertical trace being perpendicular to its vertical projection (Art. 51). A line through the given point, having the same direction as the required horizontal trace, will lie in the required plane and will pierce V in a point of the required vertical trace. The vertical trace can then be drawn through this point perpendicular to the vertical projection of the given line. Through the point where the vertical trace intersects the ground line, the horizontal trace can be drawn perpendicular to the horizontal projection of the line.

Construction. ad, perpendicular to bc, and a′d′, parallel to the ground line, are the projections of a line parallel to the required horizontal trace. This line pierces V at d′. Through d′, draw t′T perpendicular to b′c′. Through T, draw Tt perpendicular to bc. Then t′T and Tt are the vertical and horizontal traces of the required plane.

59. Problems.

1. Let the problem be solved when the point and line are both in the third angle.

2. Through a given point on a line, pass a plane perpendicular to the line.

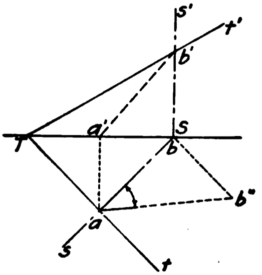
3. Through a given point, pass a plane perpendicular to a line which is parallel to H and oblique to V.

4. Through a point on G. L., pass a plane perpendicular to a given oblique line.

60. To find the angle which a plane, given by its traces, makes with the horizontal plane of projection.

Let T, Fig. 43, be the given plane.

Fig. 43



bab″ angle plane T makes with H

Analysis. A plane passed perpendicular to the horizontal trace of the given plane will cut a line from this plane and also from H. The angle between these lines is the measure of the required angle, since each line is perpendicular to the horizontal trace of the given plane.

Construction. sS and s′S are the traces of a plane perpendicular to tT. The plane S cuts the line AB from T and sS from H. The required angle Sab″ is found by revolving into H about sS as an axis.

61. Problems.

1. Find the angle which a given plane makes with V.

2. Take a plane in the third angle and find the angles which it makes with the planes of projection.

3. Given the horizontal trace and the angle which a plane makes with H, to construct the vertical trace.

4. Given the vertical trace and the angle which a plane makes with V, to construct the horizontal trace.

5. Given the vertical trace and the angle which a plane makes with H, to construct the horizontal trace.

6. Are the angles which a plane makes with the planes of projection complementary? What are the upper and lower limits for the sum of the angles which a plane makes with the planes of projection?

7. Find the traces of a plane which contains a given point P and makes an angle of θ° with H. (Use a cone with vertex at P, base on H and elements making the angle θ° with H.)

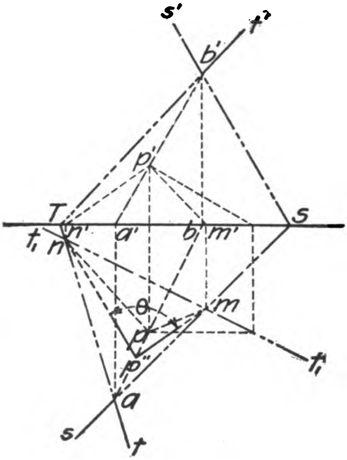
8. A line is oblique to H and V. Find the traces of a plane which makes θ° with H and contains the given line.

62. To find the angle between any two planes which are given by their traces.

Let T and S, Fig. 44, be the given planes.

Analysis. A plane passed perpendicular to the line of intersection of the given planes will cut a line from each of them which is perpendicular to the intersection. The angle between these lines is the measure of the required angle.

Fig. 44



θ angle between planes T and S

Construction. Through any point P of the intersection AB, pass a plane perpendicular to the intersection (Art. 58), t_1t_1 is the horizontal trace of this plane. pm and $p'm'$ are the projections of the line cut from S and pn and $p'n'$ the projections of the line cut from T by by the plane T_1. Revolving into H about t_1t_1 as an axis (Art. 26), the true size of the required angle is found to be θ.

63. Problems.

1. Find the angle between two planes when one is oblique to H and V and one perpendicular to H and oblique to V.

2. Find the angle between two planes when one is oblique to H and V and one perpendicular to V and oblique to H.

3. Find the angle between two planes when one is oblique to H and V and one parallel to H.

4. Find the angle between two planes when one is oblique to the ground line and one parallel to the ground line crossing the first angle.

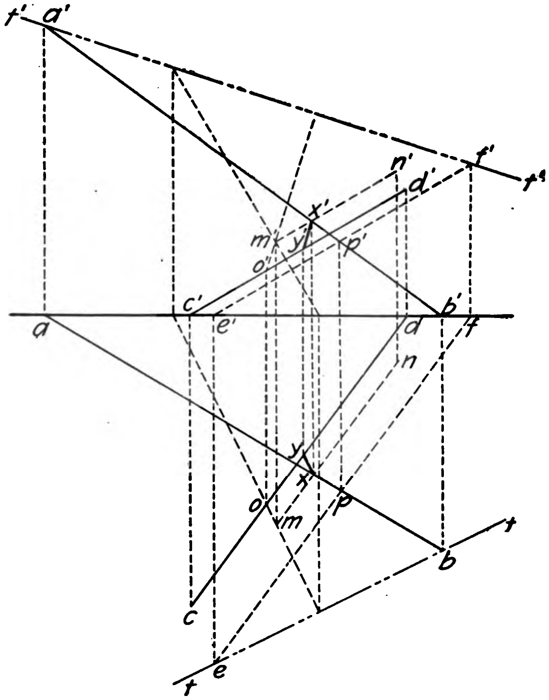
5. Find the angle between two planes when they are both parallel to the ground line.

64. To find the common perpendicular to two lines not in the same plane.

Let AB and CD, Fig.45, be the given lines.

Analysis. Pass a plane through one line parallel to the other and project the second line on this plane. This projection will be

Fig. 45



XY common perpendicular to AB and CD

parallel to the line itself, since the line is parallel with the plane. Where this projection cuts the first line, erect a perpendicular to the plane. This is the common perpendicular to the two lines.

Construction. Pass a plane through AB parallel to CD (Art. 43, Prob. 2); T is this plane. Project any point O of CD on the plane T. M is the projection of this point and MN, parallel to CD, is the projection of CD on the plane T. MN intersects AB at X, and XY, perpendicular to T, is the required line.

65. Problems.

1. Find the common perpendicular to two lines when one of them is parallel to G. L.
2. Find the common perpendicular to two lines when one of them is perpendicular to H.
3. Find the common perpendicular to two lines when one of them is perpendicular to V.
4. Find the common perpendicular to a given oblique line and the ground line.

PROBLEMS

The following problems can be solved in rectangles 10½″ × 14″; two problems can be placed on a sheet 15″ × 22″ and leave ½″ for border.

The ground line is to be parallel to the shorter sides of the rectangle.

The profile plane P is in the center of the rectangle.

Distances are given in inches.

Points are located by giving their perpendicular distances from P, V and H.

The first distance is from P.

The second distance is from V.

The third distance is from H.

Distances are measured to the right of P, in front of V and above H unless preceded by a minus sign.

1. Find the traces of a plane containing the following points: A, in the first angle; B, in the second angle; C, in the third angle.

2. Find the traces of a plane containing the following points: A, in H; B, in G. L.; C, in the first angle.

3. Find the traces of a plane containing the two parallel lines A (−4, −1¼, 3¼) B (3, −1, 3¼) and C (−4, 2, ¾) D (3, 2¼, ¾).

4. Find the length of a line joining the two points A (−3¾, 3¼, −1) and B (2, 1, 3¼):

1st. By revolving into H about the H trace of its H projecting plane

2nd. By revolving into H about the H trace of its V projecting plane.

3rd. By revolving into V about the V trace of its V projecting plane.

4th. By revolving into V about the V trace of its H projecting plane.

5. Determine the traces of a plane passing through the points A (−1½, 0, 0), B (0, 2¼, 3¼) and C (3½, 3¾, 3¼); and find the true size of the angle between the traces.

6. Given the plane t (3¼, 6¼, 0); T (−4, 0, 0,); t′ (3¼, 0, 4¼). Assume four lines in the plane passing through the point (1¾, 2½, z): 1st parallel to V; 2nd parallel to H; 3rd perpendicular to the ground line; 4th inclined 30° to V.

Revolve the plane of the lines to the left into H and show the true position of the lines.

Take the ground line 1″ above the center of the rectangle.

7. The H and V piercing points of a line are (−5, 0, 3½) and (1½, 3¾, 0) respectively. It is required to find:

1st. The projections of the line.

2nd. The projections of a line parallel to V, inclined 45° to H and intersecting the given line at the point which is nearest the ground line.

Note: Use a profile plane through a point ¾″ to the right of the origin.

8. The vertices of a triangle are at the points A (−2¾, ¾, 2½), B (−1¼, 3¾, ½), C (1¼, 2, 5¼). It is required to draw the projections of the bisectors of the three angles.

Take the axis in the plane of the triangle and 2″ above H.

9. Find the line of intersection of the planes whose traces are: t (½, 3¾, 0), T (−5¼, 0, 0), t′ (−1¼, 0, 3¾); s (½, 3¾, 0), S (1¼, 0, 0), s′ (3¼, 0, 3¾).

10. Given three points in each of three planes. It is required to find the point of intersection of the three planes.

1st (−5¼, 1¾, 0), (−1¾, 0, 0) and (−4¼, 0, 1¼).

2nd. (4½, 3¼, 0), (−1¾, 0, 0) and (−4¼, 0, 4¼).

3rd. (2, 3¼, 0), (4½, 0, 0) and (2, 0, 4¼).

11. The traces of three planes are given as follows: t(−2¾, 3½, 0), (−2¾, 3½, 0), (−3¾, 2, 0); T (−5¼, 0,0), (−¾, 0, 0), (2, 0, 0); t′ (¼, 0, 2¾), (2¾, 0, 3½), (3¾, 0, 3). Find the projections of the tetrahedron included between the three planes and H. Use an auxiliary horizontal plane 2″ above H.

12. Find the line of intersection of two planes which are parallel to the ground line and contain the points (1¼, 1, 0), (1¼, 0, 3½) and (1¼, 3½, 0), (1¼, 0, 2).

Suppose the planes to be revolved 45° to the right about a vertical axis through the point $(-\frac{3}{4}, 2, 0)$. Show the traces for the new position.

13. Given a line which is parallel to the ground line and passes through the point $(-3, 2\frac{1}{4}, 3)$. It is required to find the point in which it pierces the plane of two lines through the points $(-3\frac{1}{2}, 1, 3\frac{3}{4})$ and $(3\frac{1}{2}, -1\frac{1}{4}, 3\frac{1}{4})$, and intersecting at the point $(-\frac{1}{2}, 2, 2)$, without finding the traces of the plane.

14. Find the line which will touch the three lines: A $(-3\frac{1}{4}, 3, 1)$ B $(-1\frac{3}{4}, \frac{1}{2}, 5)$, C $(-\frac{1}{2}, \frac{1}{4}, 4)$ D $(1\frac{1}{2}, 1, \frac{1}{2})$ and E $(-1\frac{1}{4}, 1\frac{1}{2}, 4\frac{3}{4})$ F $(0, \frac{1}{2}, 1)$.

15. Through a given point take a line which touches any other two non-intersecting lines.

16. Through three given lines not situated two by two in planes, take three planes which have a common line of intersection.

17. Given a point and two planes. Required a line through the point parallel to both planes.

18. Assume four points at random. Do all the points lie in the same plane?

19. Given a plane T and a line MN which pierces the plane. Find a line in T which intersects MN and is perpendicular to it (MN is not perpendicular to the plane T.)

20. At a point of outcrop O (0, 3, 1) the strike of a body of ore is north 45° west. The "dip" (the inclination of the ore body to the horizontal) is 60°. A tunnel is driven at P $(-1\frac{1}{2}, 5\frac{1}{4}, 0)$ north 10° east on a rising 10% grade. How far will the tunnel have to be driven to reach the ore body?

Note: The "strike" means a horizontal line in the plane of the ore body.

21. From point of outcrop A, run south 30° west 100 feet to a drillhole B. The drillhole C is 100 feet from both A and B. The drill strikes ore at B at a depth of 25 feet and at C at a depth of 50 feet. Determine the dip and strike of the vein.

22. Points A, B and C are upon a sidehill which is a plane dipping west 30°. A is a point of outcrop of the vein. From A, B bears south 45° west 350 feet horizontally, while C bears

south 15° east 700 feet on the slope of the hill. At B and C, drillholes driven perpendicular to the surface each encounter the vein at 175 feet. Determine the strike and dip of the vein.

23. A borehole cuts a 6-foot core from a vein whose strike is north 75° east and whose dip is 60° to the northwest. The borehole bears north 30° east and dips 75°. Required the true thickness of the vein.

24. A horizontal tunnel cuts a vein at 1000 feet from the portal. The vein strikes north 45° west and dips 75° to the southwest. The tunnel is driven north 30° east. What will be the horizontal distance at which a tunnel on a 4% grade, driven from the same point and north 15° west, will cut the vein?

25. From A, B bears north 30° east 1500 feet; from A, C bears south 60° east 1200 feet. At A, a vertical borehole cuts a vein at 290 feet; at B, an inclined borehole bears north 75° east, dips 75° and cuts the vein at 515 feet; At C, a vertical hole cuts the vein at 375 feet. Elevation of A, 9800 feet; B, 9600 feet; C, 9900 feet. Required the strike and dip of the vein.

26. A borehole bearing north 15° east and dipping 75° intersects a vein whose strike is north 45° east. The true thickness of the vein has been found to be $3\frac{1}{2}$ feet while the core from the borehole shows 6 feet of vein matter. Find the dip of the vein.

27. Given three oil wells, A, B, C: A, 550 feet above sea level, 1450 feet deep; B, 325 feet above sea level, 975 feet deep; C, 425 feet above sea level, 1100 feet deep. Find the depth of a well to be sunk to the same oil vein at D which is 613 feet above sea level.

28. The base of a polyhedron is determined by the following four points: A $(-2\frac{3}{4}, 2\frac{3}{8}, 0)$, B $(-1\frac{1}{2}, \frac{3}{8}, 0)$, C $(-2\frac{1}{4}, 3\frac{7}{8}, 0)$, D $(\frac{3}{8}, 2\frac{3}{8}, 0)$. The V projections of the edges of the polyhedron make 60° and the H projections—30° with G. L. The upper base is parallel to H and $5''$ above it. Find the intersection of this polyhedron with the pyramid O—EFG. O$(-3, \frac{3}{8}, 5\frac{1}{4})$, E$(-1\frac{1}{8}, 4\frac{3}{4}, 0)$, F $(2\frac{1}{8}, 1\frac{1}{2}, 0)$, G $(3\frac{1}{4}, 4\frac{1}{2}, 0)$.

29. Find the intersection of a cube with a square prism when

the diagonal of the cube coincides with the axis of the prism. Edge of cube $2\frac{1}{2}''$; side of base of prism $2\frac{1}{4}''$; altitude $5''$. Axis of prism perpendicular to H, the plane of one face making 15° with V. The H projection of one top edge of the cube is perpendicular to a face of the prism.

Develop the surface of the cube showing the line of intersection.

30. Find the line of intersection of a square prism and a right square pyramid. Prism $2'' \times 2'' \times 4\frac{1}{2}''$, with edges parallel to H and 30° with V. The plane of one face makes 30° with H and the lowest edge is $\frac{3}{4}''$ above H. Base of pyramid $2\frac{1}{2}''$ square resting on H. Altitude $5''$. Side of base makes 60° with V. Axis of pyramid perpendicular to H. Axes of prism and pyramid $\frac{1}{8}''$ apart.

Note: This problem can be multiplied by changing the distance between the axes of the prism and pyramid.

31. Draw the projections of a pyramid O—ABC. $O(-\frac{3}{8}, 3\frac{7}{8}, 6)$, $A(-4\frac{3}{16}, 2\frac{1}{16}, 0)$, $B\ (3, 1, 0)$, $C\ (1\frac{1}{4}, 6, 0)$.

(1) Bisect the angles AOB, OBC, etc.

(2) Find the intersection of the pyramid with a plane; H trace—45° and V trace 30° with G. L.

Traces cut G. L. at $(-5, 0, 0)$.

(3) Find the angles which planes AOC, BOC, etc., make with H.

(4) Find the angles between the planes AOC and BOC, BOC and BOA, etc.

32. Cut a parallelogram having one side $2''$ long from the pyramid O—ABCD. $O\ (2\frac{1}{2}, \frac{5}{8}, 5\frac{1}{2})$, $A\ (-4\frac{1}{4}, 4\frac{3}{4}, 0)$, $B\ (2, 5\frac{3}{4}, 0)$, $C\ (2\frac{7}{8}, 3\frac{1}{8}, 0)$, $D\ (-\frac{3}{8}, \frac{5}{8}, 0)$.

33. Given two planes, one of which makes an angle of 30° with H and passes through the points $(4\frac{1}{4}, 0, 0)$ and $(\frac{1}{4}, 3\frac{1}{2}, 0)$; the traces of the other plane intersect at $(-1\frac{1}{4}, 0, 0)$ and make angles of 15° and 60° with the ground line in H and V respectively.

Find the angle between the two planes:

1st. By the usual method of finding the angle between two planes.

2nd. By means of a perpendicular to each plane through $(-1\frac{3}{4}, 2, 2\frac{1}{4})$.

34. Given a plane which passes through the point $(2\frac{3}{4}, 3\frac{1}{4}, 0)$ and intersects the ground line $1\frac{1}{4}''$ from the origin, the V trace making an angle of 135° with G. L. Draw the projections of a circle which lies in the plane and is tangent to the V trace.

Diameter of circle 5″. Center at the point $(-\frac{1}{4}, y, z)$.

Find the major and minor axis in each projection.

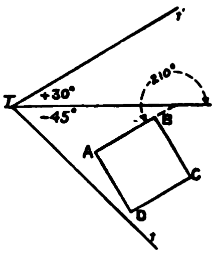
Find the tangent point.

Show how other points may be found.

35. Given a plane which passes through the points M $(1\frac{1}{4}, 0, 4\frac{1}{4})$ and T $(-2\frac{3}{4}, 0, 0)$, the H trace making an angle of − 60° with the ground line. Draw the projections of a line contained in the given plane, passing through the point M and making an angle of 45° with H. This line is the projection of another line on the plane T. This second line passes through the point M and makes an angle of 30° with the given plane. Draw its projections.

36. The shortest line between the lines A $(-\frac{1}{2}, \frac{1}{4}, 3\frac{1}{2})$ B $(\frac{3}{4}, 3\frac{3}{4}, \frac{3}{4})$ and C $(-2\frac{1}{4}, 1\frac{3}{4}, 2\frac{1}{2})$ D $(\frac{3}{4}, 1, \frac{1}{2})$ is the axis of a right rectangular prism. The diagonals of the upper and lower bases coincide with AB and CD respectively. The shorter edge of the base is $1\frac{3}{4}''$. Draw the projections of the prism.

37. Substitute A $(3\frac{1}{2}, 5, 2)$ B $(3\frac{1}{2}, 3, 0)$ and C $(-2, 2\frac{3}{4}, 2\frac{1}{4})$ D $(1\frac{3}{4}, 0, 3\frac{3}{4})$ in problem 36.

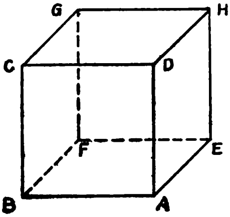


38. Given a plane T whose vertical and horizontal traces cut the ground line at the point $(-4\frac{1}{2}, 0, 0)$ and make angles of + 30° and − 45° respectively with it. Also a square ABCD on H whose side AB makes 210° with the ground line. Point B at $(1\frac{1}{4}, \frac{1}{4}, 0)$; side AB $2\frac{1}{2}''$. Required to revolve the square ABCD into plane T about Tt as an axis and find its projections in this new position.

39. Let the square in the preceding problem be the base of a

$2\frac{1}{2}''$ cube resting on the plane T. Draw the horizontal and vertical projections of the cube.

40. The horizontal and vertical traces of a plane make angles of $-45°$ and $30°$ respectively with the ground line and cut it $-4''$ from the profile plane P. A cube whose edge is $2\frac{1}{2}''$ rests on the plane in such a position that two edges AB and EF (Fig. 1) of the base make angles of 30° with the H trace. The corner B nearest H being at $(-\frac{1}{4}, y, \frac{1}{2})$. Draw the H and V projections of the cube.



(Fig. 1.)

41. Keeping the edge BA as in 40, draw the projections of the same cube having one of its corners in H.

42. Keeping the edge BA as in 40, draw the projections of the same cube having one of its corners in a plane parallel to and $1\frac{1}{4}''$ in front of V.

43. Keeping the edge BA as in 40, draw the projections of the cube having the edge BF making an angle of 30° with H.

44. Keeping the edge BA as in 40, draw the projections of the cube having the edge BF making an angle of ( )° with V.

45. Keeping the edge BA as in 40, draw the projections of the cube having the plane of its face ABFE making an angle of 30° with plane T.

46. Keeping the edge BA as in 40, draw the projections of the same cube having the plane of its face ABFE making an angle of 45° with H.

47. Keeping the corner B as in 40, draw the projections of the cube having one corner in H and another in V.

48. Find the projections of a regular hexagonal pyramid standing on a plane through A $(-1\frac{3}{4}, 1\frac{3}{4}, 2\frac{3}{4})$ parallel to the lines B $(-3\frac{1}{2}, 4\frac{3}{4}, 4)$ C $(2\frac{1}{4}, \frac{3}{4}, 2\frac{1}{4})$ and D $(-1\frac{1}{4}, \frac{1}{2}, 4\frac{3}{4})$ E $(1\frac{3}{4}, 4, \frac{1}{2})$. One side of the base lies in the line of intersection of the plane of the base and a plane through A perpendicular to the line BC. Vertex at the point $(2\frac{3}{4}, 4\frac{3}{4}, 5\frac{1}{2})$.

Find where BC and DE pierce the pyramid.

49. Substitute regular trianglar pyramid for regular hexagonal pyramid in 48.

50. Substitute regular square pyramid for regular hexagonal pyramid in 48.

51. Substitute regular pentagonal pyramid for regular hexagonal pyramid in 48.

52. Substitute a right circular cone having its base inscribed in the hexagonal base of the pyramid in 48.

53. The line A (—2¾, 0, 1½) B (—¼, 2½, 0) coincides with the edge of a cube having one corner in H and another corner in V. Edge of cube 3″. Draw the projections of the cube.

54. Draw the projections of the cube when one edge coincides with AB (as given in problem 53), one corner is at A and the corner at the other end of the diagonal through A also lies in V.

55. Draw the projections of the cube when one edge coincides with AB (as given in problem 53), one corner is at A and the corner at the other end of the diagonal of a face from A also lies in V.

56. The same as problem 53 when one corner lies at B and some other corner is also in H, but no corner in V.

57. Same as 53 when neither A nor B is a corner of the cube.

58. C (—1¼, 1¾, 1½) E (1½, 3½, 4¼) is the diagonal of a cube, One corner of the cube is in a plane whose traces make angles of —45° and 60° with the ground line in H and V respectively and intersect at a point —3¾″ from P. Draw the projections of the cube.

59. C (—3¼, 3, 1¼) E (1¼, 3, 1¼) is a diagonal of a cube. One corner is in H. Draw the projections of the cube.

60. The same as problem 59, but having a corner of the cube in V in the place of in H.

61. Given the three lines A (—¼, 3, 1) B (—2¾, 4¼, 2) A (—¼, 3, 1) C (1¾, 5½, 2) and A (¼, 3, 1) D (¼, 0, 1½). Assume on the lines, in the order named, the distances 1½″, ½″ and 1½″ from A and through each point pass a plane perpendicular to the line which contains the point.

Draw the projections of the triangular pyramid formed by the intersection of the three planes with each other and with H.

Use a horizontal plane 3″ above H for finding a second line in each face.

62. Find the projections of three spheres, each tangent to the other two and having the following radii: 1½″, 2″, 2¼″. Construct the projections of a fourth sphere with radius 1″ which will be tangent to the other three.

The plane of the centers of any three spheres is not to be parallel or perpendicular to either H or V.

63. Draw the projections of a tetrahedron two edges of which are 3″, two 4″ and two 5″. No edge is to be parallel to either H or V.

64. Draw the projections of a tetrahedron the sides of whose base are 3¼″, 3¾″ and 4″ and the planes of the faces make angles of 45°, 60° and 75° with the plane of the base. No two of the corners are to be the same distance from H.

65. Given an acute angled triangle on H. Find a point in space which joined with the vertices of the triangle will form a tri-rectangular tetrahedron.

66. Draw the projections of an equilateral triangle having given both projections of one side and the direction of the horizontal projection of one of the other sides.

67. Draw the projections of a 3″ cube having given the directions of the horizontal projections of three concurrent edges.

68. Having given the horizontal projections of two adjacent edges of a cube and the vertical projections of the corner in which these edges meet, to draw the projections of the cube.

69. Having giving the two projections ab and a′b′ of one edge of a cube and the direction ac of the horizontal projection of another edge. Draw the projections of the cube.

70. Having given the directions of the projections of three concurrent edges of a parallelopiped and their lengths, construct the projections of the parallelopiped.

71. Having given the directions of the projections of three concurrent edges of a parallelopiped, also the direction and length of the diagonal through the same corner, construct the projections of the parallelopiped.

72. Draw the projections of a rectangular parallelopiped, having giving one corner (a, a′), the horizontal projection ab of one edge, and the directions ac and ad of the horizontal projections of two other edges.

73. Given two lines A and B in space which do not intersect. Draw a third line which makes 30° with A and 45° with B and intersects them both.

74. Construct the traces of a plane which passes through a given point and which makes given angles with the planes of projection. Between what limits may the sum of these angles vary?

75. Given a line which intersects the ground line. Find the traces of a plane which will contain the given line and in such a position that the given line bisects the angle between the traces.

76. Draw the projections of a regular tetrahedron having given the projections (ab, a′b′) of one edge and the direction (ac) of the horizontal projection of another edge.

77. Given two points O and P in the first angle. Find the projections of a ray of light which emanates from O and after being reflected from both H and V passes through P.

78. Find the projections of a ray of light which emanates from a given point and is reflected from H, V and a profile plane.

CHAPTER III

CURVED LINES AND SURFACES

GENERATION AND CLASSIFICATION OF LINES

66. A line is the path of a point moving according to some law.

Lines are of two general classes:

I. **Straight Lines** or **Right Lines,** in which a point moves always in the same direction.

II. **Curved Lines,** in which a point moves so as to change its direction continually.

Curved lines are of two kinds:

I. **Curves of Single Curvature** or **Plane Curves,** in which all positions of the moving point lie in the same plane. Examples, circle, ellipse.

II. **Curves of Double Curvature,** in which all positions of the moving point do not lie in the same plane. Example, the edge of an ordinary screw thread. The line of intersection of two curved surfaces is usually of this form.

OUTLINE

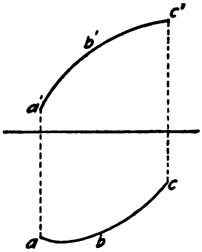
- Lines
  - I Straight or Right Lines
  - II Curved Lines
    - I Curves of Single Curvature or Plane Curves
      - Circle
      - Ellipse
      - Parabola
      - Hyperbola
    - II Curves of Double Curvature
      - Helix

PROJECTIONS OF CURVES

67. The cylinder formed by the horizontal projecting lines (Art. 6) of the points of a curve is the **horizontal projecting**

cylinder of the curve and its intersection with the horizontal plane is the **horizontal projection** of the curve. Likewise, the cylinder formed by the vertical projecting lines of the points of a curve is the **vertical projecting cylinder** of the curve and its intersection with the vertical plane is the **vertical projection** of the curve, Fig. **45.**

Fig. 45



Projections of a curve

The two projections of a curve will, in general, determine its position in space, for the projecting cylinders intersect in the only curve which can have the given projections.

If the plane of a curve of single curvature is perpendicular to the horizontal plane, the horizontal projection of the curve will be a straight line.

Likewise, if the plane of the curve is perpendicular to the vertical plane, the vertical projection will be a straight line.

If the plane of the curve is perpendicular to the ground line, both projections will be straight lines perpendicular to the ground line and the curve will be undetermined.

If the plane of the curve is parallel to the horizontal plane, the horizontal projection will be equal to the curve.

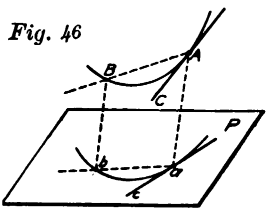
Likewise, if the plane of the curve is parallel to the vertical plane, the vertical projection will be equal to the curve.

The projections of a curve of double curvature are always curved lines.

TANGENTS AND NORMALS TO LINES

68. If in a secant line AB, Fig. 46, the point A be kept fixed and the point B moved along the curve until it coincides with A, **the secant AB becomes a tangent to the curve at the point A.**

Fig. 46



Tangent to a curve

Two curves are said to be tangent to each other at a point when they have **a common tangent at that point.**

If a straight line is tangent to a

plane curve, the tangent will lie in the plane of the curve. This is evident since the secant is in the plane of the curve, and as it moves about the point A it remains in this plane.

Two straight lines tangent to each other will coincide. In this case the secant of Fig. 46 coincides with the given line.

69. If two lines are tangent to each other in space, their projections on the same plane will be tangent. Let Fig. 46 represent a curve in space with its secant AB. Let these lines be projected upon any plane. Then the projections of the points A and B will approach each other as the points A and B in space approach each other. When the secant AB becomes a tangent in space, the points A and B coincide and their projections will also coincide in the only point common to the projections of the straight line and curve. The projections of the lines are therefore tangent to each other at this common point.

70. Normals. If a line be drawn perpendicular to the tangent at the point of tangency, it is called a normal to the curve. There are an infinite number of normals to a curve at a point on the curve. However, the normal to a plane curve is understood to be the one lying in the plane of the curve unless otherwise stated.

71. Rectification of curves. To rectify a curve means to find a straight line equal in length to the curve. This is accomplished approximately by dividing the curve into a number of small arcs, so small that for all practical purposes the chords of these arcs may be taken as equal in length to the arcs themselves. These small chords are then laid off one after another along a straight line. The part of the line thus covered is the rectified curve.

CURVES OF SINGLE CURVATURE

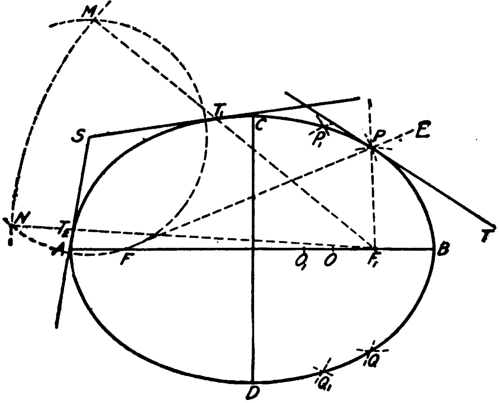
72. The simplest plane curves are the circle, ellipse, parabola and hyperbola.

A Circle is the path of a point moving in a plane so that its distance from a given point is constant.

73. An ellipse is the path of a point moving in a plane so that the sum of its distances from two fixed points is constant.

Construction. Let F and F_1, Fig. 47, be the two fixed points and let the sum of the distances from these points to any point on the curve be equal to the line AB. Select any point, as O, on the line AB. With F as center and AO as radius, strike an arc; also with F_1 as center and BO as radius, strike an arc cutting the first arc in the points P and Q. These are points on the curve since $FP+F_1P=AB$ and $FQ+F_1Q=AB$. Selecting any other point on AB, as O_1, and going through the construction as before, two other points P_1 and Q_1 are located. By continuing this process enough points can be located so that the curve can be drawn.

Fig. 47



Ellipse and tangents

The points F and F_1 are the **foci** of the ellipse. AB is the **major axis.** The **minor axis** CD is perpendicular to the major axis at its middle point. The intersection of the axes is the **center** of the ellipse. Lines drawn from any point on the curve, as P, to the foci F and F_1 are called **focal radii.**

If the axes of an ellipse are given, the foci can be found by taking one-half the major axis as a radius and with the end of the minor axis as a center, cut the major axis in two points which are the foci.

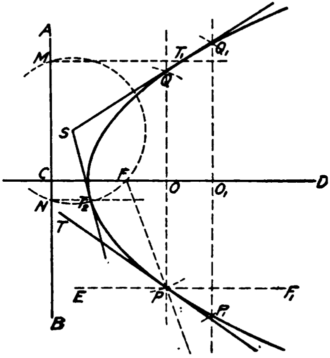
To draw a tangent to the ellipse at a point on the curve. Let P, Fig. 47, be the point on the curve. Draw the focal radii PF and PF_1. The line PT which bisects the angle F_1PE is the required tangent. (Wood's Co-ordinate Geometry, Art. 108A.) This method applies to all conics.

To draw a tangent to the ellipse from a point without the curve. Let S, Fig. 47, be the point without the curve. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points M and N. F_1M and F_1N intersect the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents. (Wood's Co-ordinate Geometry, Art. 108B.) This method applies to all conics.

74. A parabola is the path of a point moving in a plane so that its distance from a given point is always equal to its distance from a given straight line.

Construction. Let F, Fig. 48, be the fixed point and AB the given straight line. Draw a line CD from F perpendicular to AB. Through any point, as O, on CD, draw a line parallel with AB. With F as center and CO as radius strike an arc cutting this line in the two points P and Q. P and Q are points on the parabola, since FP and FQ are each equal to the distance of P and Q from AB. Selecting some other point on the axis and going through a similar construction, two other points of the curve are located. By continuing this process, enough points can be located so that the curve can be drawn.

Fig. 48



Parabola and tangents

The point F is the **focus,** the line AB the **directrix** and the line CD the **axis** of the parabola.

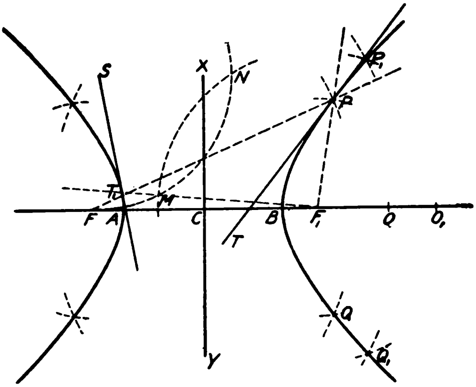
To draw a tangent to the parabola at a point on the curve. Let P, Fig. 48, be the point on the curve. Draw the focal radii FP and F_1P. (To make this construction similar to that for the tangent to the ellipse, it is necessary to consider one focus of the parabola at an infinite distance on the axis.) The line PT which bisects the angle FPE is the required tangent.

To draw a tangent to the parabola from a point without the curve. Let S, Fig. 48, be the point without the curve. With S as center and SF as radius, strike an arc cutting the directrix in the points M and N. MF_1 and NF_1, parallel with the axis CD of the parabola, cut the curve in the points of tangency T_1 and T_2. ST_1 and ST_2 are the required tangents.

75. The hyperbola is the path of a point moving in a plane so that the difference of its distances from two fixed points is constant.

Construction. Let F and F_1, Fig. 49, be the two fixed points

Fig. 49



Hyperbola and tangents

and let the difference of the distances from these points to any point on the curve be equal to the line AB. Select any point, as O, on the line AB extended. With F as center and AO as radius, strike an arc; also with F_1 as center and BO as radius, strike an arc cutting the first arc in the points P and Q. These are points on the curve since $FP - F_1P = AB$ and $FQ - F_1Q = AB$. (If F_1 be taken as center and AO as radius and F as center and BO as radius, the points located will be on the other branch of the hyperbola.) Selecting any other point on AB, as O_1, and going through the construction as before, two other points P_1 and Q_1 are located. By continuing this process enough points can be located so that the curve can be drawn.

The points F and F_1 are the **foci** of the hyperbola. The point C, midway between the foci F and F_1, is the **center** of the curve. The line AB is the **transverse axis.** The line XY, perpendicular to AB at its middle point, is the **indefinite conjugate axis** of the curve.

To draw a tangent to the hyperbola at a point on the curve. Let P, Fig. 49, be the point on the curve. Draw the focal radii FP and F_1P. The line PT which bisects the angle F_1PF is the required tangent.

To draw a tangent to the hyperbola from a point without the curve. Let S, Fig. 49, be the point without the curve. With F_1 as center and AB as radius, strike an arc. With S as center and SF as radius, strike an arc cutting the first arc in the points M and N. F_1M and F_1N intersect the curve in the points of tangency. ST_1 is one of the tangents. The other point of tangency, where NF_1 intersects the curve, is without the limits of the drawing.

CURVES OF DOUBLE CURVATURE

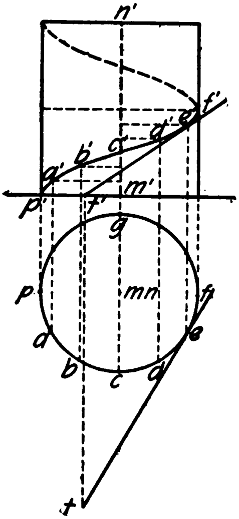
76. An ordinary helix is the path of a point moving on the surface of a cylinder of revolution so as to intersect its elements at a constant acute angle.

The axis of the cylinder is the **axis of the helix.**

Construction. Let MN, Fig. 50, be the axis of the helix and P the generating point.

Suppose that for one complete turn around the axis the generating point moves through a vertical distance m′n′. This distance is the **pitch** of the helix.

Fig. 50



Helix and tangent

Since the curve is on the surface of a cylinder of revolution, its projection on a plane perpendicular to the axis will be the circle pcfg.

To draw the vertical projection of the helix, divide the circle pcfg into any number of equal parts, as twelve, and m′n′ into the same number of parts. Draw horizontal lines through the points of division of m′n′. Since the motions of the point around and along the axis are both uniform, the point in making one-twelfth of a complete turn around the axis will rise one-twelfth of the distance m′n′. a is the horizontal and a′ the vertical projection of the point after making one-twelfth of a turn. In the same manner, the points b′, c′, d′, etc., are found. The vertical projection of the curve is drawn through the vertical projections of these points.

77. Since the curve cuts all the elements of the cylinder at the same angle, it is evident that the helix will become a straight line if the cylinder is opened along an element and the surface rolled out into a plane. From this position it is seen that the hypothenuse of a right triangle will form a helix if the altitude of the triangle coincides with an element of a right circular cylinder and the base of the triangle is wound around the base of the cylinder.

To construct a tangent to a helix at a point on the curve. Let E, Fig. 50, be the point at which the tangent is to be drawn. Let the triangle referred to in the above paragraph as being wound around the cylinder, be unrolled as far as the point of tangency E. The hypothenuse of the triangle in this position is tangent to the curve, since it touches the curve at the point E and makes the same angle with the horizontal plane as the curve. The base et of the triangle is the horizontal projection of the tangent, and is equal in length to the arc ep of the circle pcfg. Therefore, to get the projections of the tangent, draw its horizontal projection et tangent to the circle pcfg at the point e. On this line lay off from e the true length of the arc ep of the circle, Art. 71. This will give the point t where the tangent pierces the horizontal plane. The vertical projection of this piercing point is at t′ in the ground line, and this joined with e′ gives the vertical projection e′t′ of the tangent.

GENERATION AND CLASSIFICATION OF SURFACES

78. A surface is the path of a line moving according to some law.

The moving line is the **generatrix,** and its different positions are the **elements** of the surface.

Surfaces are of two general classes:

I. **Ruled surfaces,** which can be generated by straight lines. Examples, cone, cylinder.

II. **Double curved surfaces,** which can only be generated by curved lines. Example, sphere.

Ruled surfaces are of three kinds:

I. **Plane surfaces.**

II. **Single curved surfaces,** which can be rolled into a plane without undergoing distortion. Examples, cylinder, cone.

III. **Warped surfaces,** which cannot be rolled into a plane without undergoing distortion. Example, the surface of a screw thread.

Single curved surfaces are of three kinds:

I. **Cylinders,** in which the rectilinear elements are parallel.

II. **Cones,** in which the rectilinear elements intersect in a point.

III. **Convolutes,** in which the rectilinear elements are tangent to a curve of double curvature. Example, the surface having tangents to a helix for its elements.

OUTLINE

Surfaces
- I Ruled
  - I Plane
  - II Single curved
    - Cylinders
    - Cones
    - Convolutes
  - III Warped
    - Helicoid
    - Hyperbolic Paraboloid
    - Conoid
    - Cylindroid
    - Hyperboloid of Revolution of one Sheet
- II Double curved
  - Sphere
  - Ellipsoid of Revolution
  - Paraboloid of Revolution
  - Hyperboloid of Revolution of two Sheets

SURFACES OF REVOLUTION

79. A surface of revolution is the path of a line revolving about a straight line as an axis.

It is evident that the intersection of this surface with a plane perpendicular to the axis is a circle.

A surface of revolution may be generated by a circle having its center moving along a straight line, its different positions in parallel planes and its radius changing according to a given law.

The intersection of the surface with a plane containing the axis is a **meridian line,** and the plane is a **meridian plane.** All meridian lines of the same surface are identical. Any surface of revolution may be generated by revolving its meridian line about its axis.

OUTLINE

- Surfaces of Revolution
  - Ruled
    - Single curved
      - Right circular cylinder
      - Right circular cone
    - Warped
      - Hyperboloid of Revolution of one Sheet
  - Double curved
    - Sphere
    - Ellipsoid of Revolution
      - Prolate
      - Oblate
    - Paraboloid of Revolution
    - Hyperboloid of Revolution of two Sheets

80. If two surfaces of revolution have a common axis, what will be their line of intersection?

What must be the relative position of the axis and generatrix to form a cylinder of revolution? A cone of revolution?

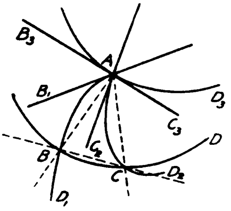
The cylinder and the cone are the only single curved surfaces of revolution.

The hyperboloid of revolution of one sheet is the only warped surface of revolution.

TANGENT PLANES TO SURFACES. NORMAL LINES AND PLANES

81. Let AD_1 and AD_2, Fig. 51, be any two intersecting curves on a surface, and BCD a curve which if moved along the curves AD_1 and AD_2 will generate the given surface. The curve BCD may vary in form as it moves. The secants AB, BC and AC lie in one plane. As the curve BCD moves toward A, the points of intersection B and C will travel along the curves AD_1 and AD_2 until the secants AB and AC become the tangents AB_1 and AC_2 at the point **A**. The curve AD_3 is the position of the moving curve BCD when the points B and C reach A. In this position, the secant BC becomes the

Fig. 51



Tangent plane to a surface

tangent B_3C_3. The tangents AB_1, AC_2 and B_3C_3 all lie in one plane which is called the **tangent plane** to the **surface** at the point A. The point A is the **point of contact.**

Since AD_1 and AD_2 are any curves on the surface, it follows that, in general, the tangent at A to every curve of the surface through this point will lie in the tangent plane. Therefore, **the tangent plane will contain all straight lines tangent to lines of the surface at the point of contact.**

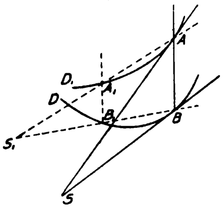
If any plane be passed through the point of contact, it will cut a straight line from the tangent plane and a line from the surface, and these lines will be tangent to each other.

Since two intersecting lines will determine a plane, it follows that **a plane tangent to a surface at a given point is determined by two straight lines tangent at this point to two lines of the surface.**

In any ruled surface, the rectilinear element through the point of contact lies in the tangent plane; for the rectilinear element and its tangent coincide (Art. 68).

82. A tangent plane to a single curved surface is tangent all along a rectilinear element. In Fig. 52, let D be the curve of the base of a cylinder and AB and A_1B_1 any two rectilinear elements of the surface. From any point A of one element draw on the surface a curve D_1 which cuts the other element at A_1. The chords AA_1 and BB_1 lie in the same plane and will intersect at some point as S_1. Now if the plane ABB_1A_1 be revolved about AB as an axis, the point A_1 will approach A along the curve D_1, and B_1 will approach B along the curve D. When the point A_1 reaches A, the secant AS_1 becomes the tangent AS to the curve D_1, and at the same time B_1 reaches B, the secant S_1B becoming the tangent SB to the curve D. The plane SAB is tangent to the cylinder at the point B (Art. 81), since it contains the tangent SB to the curve D, and also the

Fig 52



Tangent plane to single curved surface rectilinear element AB. This plane is also tangent to the cylinder at A, containing the tangent SA to the curve D, and the element AB. Since A was taken as any point on the element AB, the plane SAB must be tangent to the surface at every point of this element.

The same demonstration applies to the tangent plane to a cone.

In the case of the convolute, the element A_1B_1 is a curve instead of a straight line. The curve will change its form as the secant plane is revolved, finally becoming a straight line coinciding with the axis AB. Otherwise the demonstration given above also applies to the convolute.

Since the plane is tangent all along the element, the intersection of the tangent plane with the plane of the base will be a straight line tangent to the curve which represents the base (Art. 81). For example, if the base of a single curved surface is on H, the horizontal trace of the tangent plane will be tangent to the base.

83. Tangent planes to warped surfaces. Since warped surfaces have straight line elements, the tangent plane to the surface at any point will contain the rectilinear element through that point (Art. 81). If any curve of the surface be drawn through the point of contact, the straight line tangent to the curve at this point and the rectilinear element through the point will determine the tangent plane (Art. 81).

If the warped surface is of such a form that there are two rectilinear elements through each point of the surface, then the tangent plane will be determined by the two elements through the point of contact.

Although the tangent plane to a warped surface contains a rectilinear element of the surface, it is, in general, tangent at only one point of this element. The tangent plane usually intersects the surface. These principles will be better understood when the tangent planes to some of the surfaces are drawn.

84. Tangent planes to double curved surfaces. Through the point on the surface at which the tangent plane is to be passed, draw any two curves of the surface. The straight lines tangent to these curves at their point of intersection will determine the tangent plane. If care is taken to select simple curves on the surface in simple positions with reference to the planes of projection, the tangents to these can be easily drawn.

In the case of a **double curved surface of revolution,** the simplest curves are usually the meridian curve and a circle which lies in a plane perpendicular to the axis of the surface.

85. Normals. A straight line perpendicular to the tangent plane at the point of contact is a **normal to the surface.**

Any plane containing a normal line is a **normal plane** to the surface.

SINGLE CURVED SURFACES

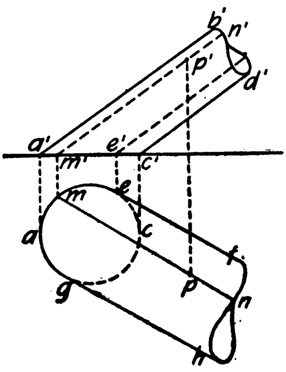
CYLINDERS

86. A cylinder is the path of a straight line moving along a curve and remaining parallel to another straight line.

The curve is the **directrix** and the moving line is the **rectilinear generatrix.**

The intersection of the cylinder with the horizontal plane is usually taken as the **base.** Any plane section may be considered to be the base. If this base has a center, the straight line through it parallel to the rectilinear elements is the **axis** of the cylinder.

Fig. 53



Cylinder

A right cylinder is one in which the rectilinear elements are perpendicular to the plane of the base.

Any section taken perpendicular to the elements is a **right section.**

If a cylinder is intersected by a plane parallel to the rectilinear generatrix, it will cut rectilinear elements from the surface.

87. To represent the surface. A cylinder is usually represented by the projections of its base and its extreme elements.

In Fig. 53, aecg is the base and EF

the rectilinear generatrix. The extreme elements in the horizontal view are tangent to the base and parallel to the horizontal projection of the rectilinear generatrix. The extreme elements in the vertical view are drawn through the extreme points of the vertical projection of the base and parallel to the vertical projection of the rectilinear generatrix.

Since the surface is indefinite in extent, it can be represented with the upper end broken off unless some definite portion of the surface is taken for a particular purpose.

88. To represent a rectilinear element of the surface. Through any point M of the base draw a straight line MN parallel to the rectilinear generatrix; this will be an element of the surface.

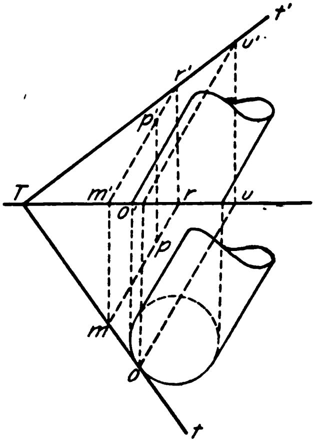
To represent a point of a surface. Locate a rectilinear element, as MN, and then take any point P of this element.

89. To pass a plane through a given point and tangent to a cylinder.

The tangent plane to a cylinder is tangent all along a rectilinear element.

The intersection of the tangent plane with the plane of the base is tangent to the base (Art. 82).

Fig. 54



Tangent plane to a cylinder

Let the cylinder be given as in Fig. 54, and let P be the given point.

Since the tangent plane must contain a rectilinear element of the cylinder, a straight line through P and parallel to the rectilinear elements will lie in the tangent plane and will pierce H in M, a point of the horizontal trace of this plane. A tangent to the base through M will be the horizontal trace of the required tangent plane. An element through the point of tangency of the horizontal trace will be the element of contact.

This element and the line through P parallel to it will determine the tangent plane.

There can be as many tangent planes to the cylinder through the given point as there can be lines drawn tangent to the base of the cylinder through the point where the auxiliary line pierces the plane of the base.

90. Problems.

1. Through a given point without the surface, pass a plane tangent to a right circular cylinder.
2. Pass a plane tangent to a cylinder at a given point on the surface.
3. Pass a plane tangent to a cylinder and parallel to a given straight line. (A plane containing the given line and a line parallel to the elements of the cylinder will be parallel to the required tangent plane.)

CONES

91. A cone is the path of a straight line moving along a curve and passing through a fixed point.

The curve is the **directrix,** the fixed point the **vertex** and the moving line the **rectilinear generatrix** of the cone.

The generatrix being indefinite in length, will generate two parts of the surface on the opposite sides of the vertex called the **upper and lower sheets.**

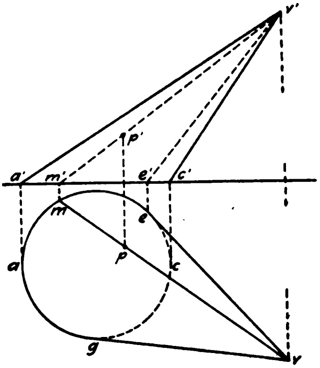
The intersection of the cone with the horizontal plane is usually taken as the **base.** Any plane section may be considered to be the base.

A right cone is one in which all the rectilinear elements make the same angle with a straight line through the vertex called the **axis** of the cone.

Any plane passing through the vertex of a cone will cut rectilinear elements from the surface.

92. To represent the surface. A cone is usually represented by the projections of its base, vertex and extreme elements.

Fig. 55



Cone

In Fig. 55, aecg is the base and V the vertex of the cone. The extreme elements in the horizontal view are tangent to the base and pass through the horizontal projection of the vertex. The extreme elements in the vertical view are drawn through the extreme points of the vertical projection of the base and through the vertical projection of the vertex.

93. To represent a rectilinear element of the surface.

Through any point M of the base, draw a straight line through the vertex V; this will be an element of the surface.

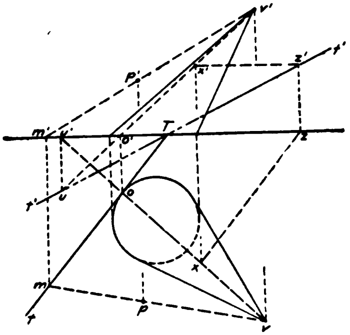
To represent a point of the surface. Locate a rectilinear element, as MV, and take any point P of this element.

94. To pass a plane through a given point and tangent to a cone.

The tangent plane to a cone is tangent all along a retilinear element.

The intersection of the tangent plane with the plane of the base is tangent to the base (Art 82).

Fig. 56



Tangent plane to a cone

Let the cone be given as in Fig. 56, and let P be the given point.

Since the required plane must contain a rectilinear element of the cone, and therefore the vertex, a straight line joining the vertex with the given point P will lie in the tangent plane and pierce H at M, a point of the horizontal trace of this plane. A tangent to the base through this point will be the horizontal trace of the required tangent plane. An element through the point of tangency will be the element of contact. This element and the horizontal trace will determine the tangent plane.

If more than one tangent to the base is possible, there will be more than one solution to the problem.

95. Problems.

1. Pass a plane tangent to a cone of revolution through a point without the surface.

2. Pass a plane tangent to a cone through a given point on the surface.

3. Pass a plane tangent to a cone and parallel to a given straight line. (A line through the vertex parallel to the given line will lie in the required tangent plane.)

CONVOLUTES

96. A convolute is the path of a straight line moving along and remaining tangent to a curve of double curvature such as the helix. The surface is represented by drawing the projections of the curvilinear directrix, an occasional element and the intersection of the surface with the horizontal plane. If the curvilinear directrix is a helix with axis perpendicular to the horizontal plane, the base of the surface will be the involute of the circle which represents the horizontal projection of the helix.

This surface is of very little practical importance, so no further discussion of it will be given. For a complete discussion of it, see Salmon's Geometry of Three Dimensions, Art. 349.

WARPED SURFACES

97. There is a great variety of warped surfaces, differing from each other in their mode of generation and properties. Only a few of the more common ones will be discussed.

There are several kinds of warped surfaces formed by moving a straight line to touch two other lines, straight or curved, and remaining parallel to a given plane, called **a plane directer.**

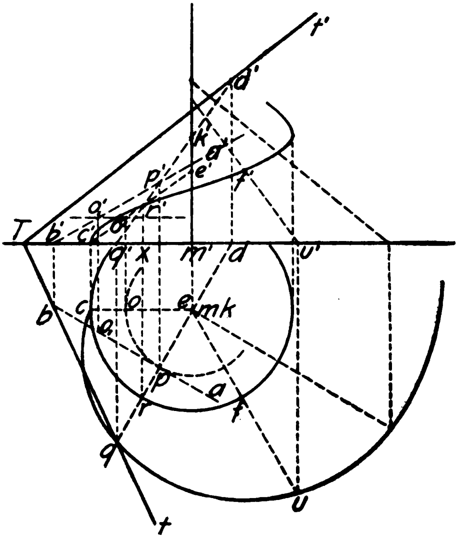
Warped surfaces are usually represented by drawing the projections of one or more curves of the surface and a few of the rectilinear elements. The projections of the directrices are often given.

HELICOIDS

98. An ordinary **helicoid** is the path of a straight line moving along a helix and its axis, and remaining at a given angle with the axis.

The axis of the helix is the **axis** of the surface.

Fig. 57



Helicoid and tangent plane

In Fig. 57, MK is the axis perpendicular to the horizontal plane, and CE is the generating line in the initial position parallel to the vertical plane. The point C will generate the helix CRF which must first be constructed (Art. 76).

To represent a rectilinear element of the surface. Connect any point R of the helix with a point K on the axis, making EK equal to the distance of the point R above the horizontal plane. Hence in the vertical projection, e'k' is equal to r'x Then r'k' is the vertical and rk the horizontal projection of a rectilinear element.

To represent a point of the surface. Locate a rectilinear element, as RK, and take any point of this element as P.

To find the base of the surface in the horizontal plane, produce the rectilinear elements until they cut this plane. In Fig. 57, cqu is the base of the surface.

When the generatrix is perpendicular to the axis, the surface is a **right helicoid,** as in the square screw thread. When the generatrix is at an acute angle with the axis, the surface is an **oblique helicoid,** as in the triangular screw thread.

99. To pass a plane tangent to a helicoid at a point on the surface.

Let the surface be given as in Fig. 57 and let P be the given point represented as in Art. 98.

Analysis. Construct a helix passing through the point P and lying on the surface of the helicoid. The tangent to the helix at the point P and the rectilinear element of the surface through this point will determine the tangent plane.

Construction. The helix through the point P is constructed by the method of Art. 76. The pitch of this helix must be the same as the pitch of the helix CRF which represents the helicoid. The arc op is the horizontal projection of part of this helix. The straight line po_1 is made equal to the arc po and o_1' is located, it being the same distance above the ground line as o'. Then ab and a'b' are the projections of the tangent to the helix at the point P (Art. 77). qd and q'd' are the projections of the element of the helicoid passing through P (Art. 98). The plane T, determined by the two lines AB and QD, is the required tangent plane.

HYPERBOLOIDS OF REVOLUTION OF ONE SHEET

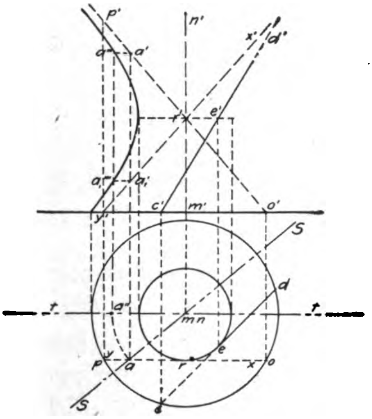
100. An hyperboloid of revolution of one sheet is the path of a straight line revolving about another straight line as an axis, the generatrix and axis not being in the same plane.

From the nature of the motion it is evident that no two positions of the generatrix can be brought into one plane without distortion, therefore the surface is warped (Art. 78).

This is the only warped surface of revolution.

101. To represent the surface. In Fig. 58, MN is the axis and OP the initial position of the generatrix. Each point of the generatrix OP describes a horizontal circle. The point R, nearest the axis, describes a circle of radius nr, called the **circle of the gorge.** The point O describes a circle on the horizontal plane usually considered the **base** of the surface. The base and horizontal projection of the circle of the gorge form the horizontal view of the surface.

Fig. 58



Hyperboloid of revolution of one sheet

The vertical view will be formed by the vertical projection of the meridian section parallel to the vertical plane. To find this view, take any point A on the generatrix OP and revolve it about MN as an axis until it comes into the meridian plane T at the point A″. The V projection, a‴, of the point in this position is a point in the vertical view of the surface. In the same way any number of points can be found and the vertical view determined.

102. In each position of the generatrix OP there is a radius of the circle of the gorge perpendicular to it. Since the radius is parallel to H, its H projection will be perpendicular to the H projection of the generatrix. Hence the horizontal projections of the elements of the surface are tangent to the horizontal projection of the circle of the gorge.

Therefore, **to represent an element of the surface,** draw its horizontal projection cd tangent to the horizontal projection of the circle of the gorge. The vertical projection of C is c′ and of the point of tangency E is e′. Hence c′e′d′ is the vertical projection of the element.

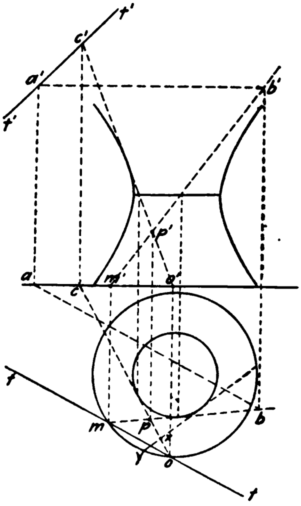
To represent a point of the surface. Locate an element and then take a point of this element.

103. In Fig. 58, draw a line XY through the point R parallel to V and making the same angle with H as the generatrix OP. For each point of XY there is a corresponding point of OP which is the same distance from H and also from MN. As OP and XY revolve about MN as an axis, these two points will generate the same circle and the two lines will therefore generate the same surface. Hence it is evident that **through every point of the surface two rectilinear elements can be drawn.**

104. To pass a plane tangent to a hyperboloid of revolution of one sheet at a point of the surface.

Let P, Fig. 59, be the given point (Art. 102).

Fig. 59



Tangent plane to hyperboloid of revolution of one sheet

The rectilinear elements through the point of contact coincide with their tangents (Art. 68), and therefore determine the tangent plane to the surface at that point.

Construction. Through P draw the rectilinear elements PM and PO (Art. 102). These elements pierce H at m and o respectively, two points which fix the horizontal trace of the required tangent plane. A line BA through any point B of the element PM and parallel to the horizontal trace tt, will pierce V in a point A of the vertical trace. The element P O pierces V at C. Joining C and A gives the vertical trace t′t′ of the required plane. Since the horizontal trace tt intersects the base, it is evident that **the tangent plane is also an intersecting**

plane. The plane T is only tangent to the surface at P since a rectilinear element through any other point of the element OP, as X, will pierce H at y, and therefore will not lie in the plane T. Hence **a tangent plane to a warped surface will contain a rectilinear element of the surface and will be tangent to the surface at but one point of that element.**

HYPERBOLIC PARABOLOIDS

105. An hyperbolic paraboloid is the path of a straight line moving along any two straight lines not in the same plane, and remaining parallel to a given plane.

The moving line is called the **generatrix,** the two fixed straight lines the **directrices** and the given plane the **plane directer.** Any position of the generatrix is called an **element** of the surface.

This surface has a second rectilinear generation in which any two rectilinear elements of the first generation may be taken as directrices with a plane directer parallel to the first directrices. It follows that **through any point of a hyperbolic paraboloid, two rectilinear elements can always be drawn.**

The intersection of the surface by a plane is an hyperbola or a parabola, hence its name.

It is evident from the nature of the motion of the generating line that the surface is warped.

To represent the surface. An hyperbolic paraboloid is usually represented by projecting the apparent contour, the two rectilinear directrices and two rectilinear elements.

To represent a rectilinear element of the surface. Let AB and CD, Fig. 60, be the rectilinear directrices and let V be the plane directer. Through any point M on the directrix AB, draw MN parallel to the plane directer V and intersecting the other directrix CD at N. mn will be the horizontal and m′ n′ the vertical projection of a rectilinear element of the surface.

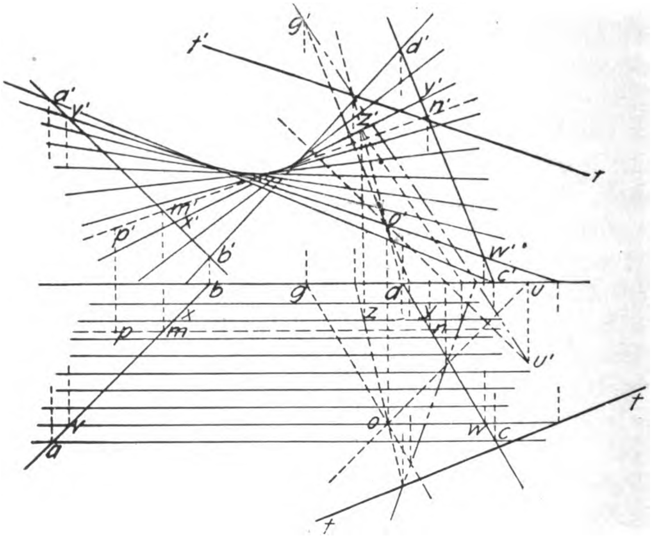
To represent a point of the surface. Represent a rectilinear element, as MN, Fig. 60, then take any point of this element, as P. p and p′ will be the projections of the required point.

106. To pass a plane tangent to the surface at a given point of the surface.

Let O, Fig. 60, be the given point.

Analysis. Through the given point pass a plane parallel to the elements of each generation. These planes will intersect the surface in the two rectilinear elements which determine the tangent plane.

Fig. 60



Tangent plane to hyperbolic paraboloid

Construction. The plane through O parallel to the elements of the first generation will cut the element VW from the surface. Through O draw OG parallel to CD and OU parallel to AB. These lines will determine a plane parallel to the elements of the second generation. This plane will intersect the element XY in Z. The line joining Z with O will be an element of the second generation. OW and OZ will therefore determine the required tangent plane T.

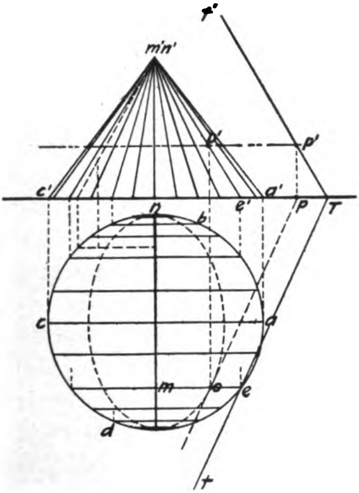
CONOIDS

107. A conoid is the path of a straight line moving along two other lines, one straight and the other curved, and remaining parallel to a given plane.

To represent the surface. A conoid is usually represented by projecting the directrices and the rectilinear elements of apparent contour.

To represent a rectilinear element of the surface. Let the circle ABCD, Fig. 61, and the straight line MN, be the given directrices and let V be the plane directer. Through any point E on the curved directrix draw EM parallel to the plane directer V and intersecting the rectilinear directrix at M. em will be the horizontal and e′m′ the vertical projection of a rectilinear element of the surface.

Fig. 61



Tangent plane to conoid

To represent a point of the surface. Represent a rectilinear element, as EM, Fig. 61, and then take any point of this element, as O. o and o′ will be the projections of a point of the surface.

108. To pass a plane tangent to the surface at a given point of the surface. Let O, Fig. 61, be the given point. The tangent plane must contain the rectilinear element EM through the given point. It must also contain the line OP tangent to the elliptical section at the same point. EM and OP will therefore determine the required tangent plane T.

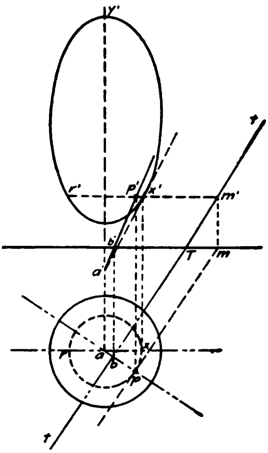
1. Represent a point on the surface of a sphere.
2. Represent a point on the surface of a prolate ellipsoid.
3. Represent a point on the surface of an oblate ellipsoid.
4. Represent a point on the surface of a paraboloid of revolution.

113. Tangent planes to double curved surfaces of revolution. The tangent plane to a surface at a point on the surface is determined by two straight lines tangent at this point to two lines of the surface. The simplest curves passing through the point on a surface of revolution are usually the meridian line and the circle which is cut out by a plane perpendicular to the axis of the surface. The tangent plane, therefore, is determined by two straight lines, one tangent to the circular section at the given point and the other tangent to the meridian section at that point.

114. To pass a plane tangent to an ellipsoid of revolution at a point on the surface.

Let the surface be given as in Fig. 63, and let P be the given point represented by the method of Art. 111.

Fig. 63



Tangent plane to ellipsoid of revolution

Analysis. A tangent to the meridian curve at the given point and a tangent to the circle at this point will determine the tangent plane (Art. 113).

Construction. prx is the horizontal and r′x′ the vertical projection of the circle of the surface through the given point P. pm is the horizontal projection and p′m′ the vertical projection of the tangent to the circle at the point P. The plane of the meridian section through P has ap for its horizontal trace. If the meridian curve through P is revolved about the axis of the surface until the plane of the curve becomes parallel to V, it will have the ellipse r′x′y′ for its vertical projection, P moving to X. At x′ draw x′a′ tangent to the ellipse x′r′y′.

It will be the vertical projection of the revolved position of the tangent to the meridian section at P. In the counter revolution of the meridian plane, X moves to P and the point A, in the axis, remains fixed. Then ap will be the horizontal and a′p′ the vertical projection of the tangent to the meridian curve at P. This tangent pierces H at b, one point of the horizontal trace of the required plane. The tangent PM is parallel to H and pierces V at m′. Through b, draw tT parallel to pm and through m′ draw t′T; then T is the required plane tangent to the surface at P.

115. Problems.

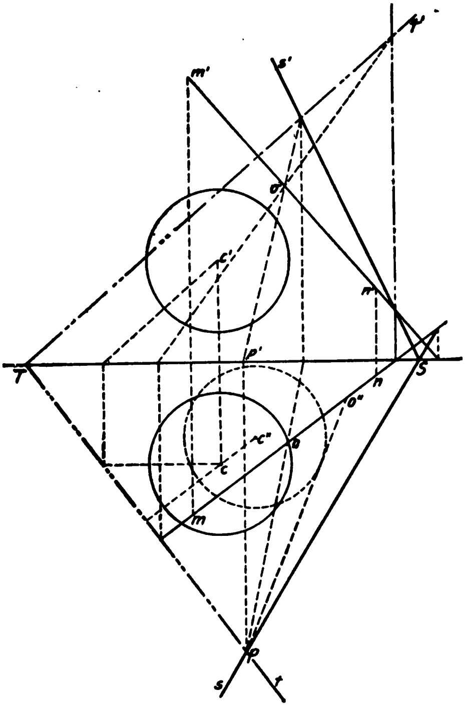
1. Pass a plane tangent to a sphere at a point of the surface.
2. Pass a plane tangent to an oblate ellipsoid at a point of the surface.
3. Pass a plane tangent to a paraboloid of revolution at a point of the surface.

116. To pass a plane through a given straight line and tangent to a sphere.

Let MN, Fig. 64, be the given straight line and C the center of the given sphere.

Analysis. Assume that the required plane is drawn through the given line and tangent to the sphere. Now if an auxiliary plane be passed

Fig. 64



Plane S contains line MN and is tangent to sphere C through the center of the sphere and perpendicular to the line MN, it will cut a point from MN, a great circle from the sphere and a line from the tangent plane which will pass through the point on MN and be tangent to the great circle cut from the sphere. Therefore, to make the construction for the tangent plane, pass a plane through the center of the sphere and perpendicular to the line MN. From the point where this plane cuts MN, draw a tangent to the great circle it cuts from the sphere. This tangent and the line MN will determine the required tangent plane.

Construction. tT and t′T are the traces of the plane which passes through C and is perpendicular to MN (Art. 57). MN pierces this plane at O. When the plane T is revolved into H about tT as an axis, O moves to o″ and C to c″. o″p is the revolved position of the tangent to the great circle cut from the sphere. op and o′p′ are the projections of this tangent after the counter revolution of the plane T. The plane S, which contains the lines MN and OP, is the required tangent plane to the sphere.

There are usually two planes which will contain a given straight line and be tangent to a sphere. The other plane is determined by MN and the other tangent to the great circle from the point O.

PROBLEMS

1. To a given oblique cone with base on H, pass a tangent plane which makes a given angle with H.

2. To a given oblique cone with base on V, pass a tangent plane which makes a given angle with V.

3. To a given oblique cylinder with base on H, pass a tangent plane which makes a given angle with H.

4. To a given oblique cylinder with base on V, pass a tangent plane which makes a given angle with V.

5. Find a plane which contains the ground line and is tangent to a given sphere. Find the angles which the plane makes with the planes of projection.

6. Pass a plane tangent to a sphere and parallel to a given plane.

7. Inscribe a sphere in a given tetrahedron.

8. Through a given straight line, pass a plane tangent to an ellipsoid of revolution.

9. Pass a plane tangent to an ellipsoid and parallel to a given plane.

10. Having given the axis of a cylinder of revolution and a tangent plane, draw the projections of the element of contact. (The tangent plane must be parallel to the axis.)

11. Through a given point, pass a plane tangent to a cylinder of revolution, having given the axis and radius of the cylinder.

12. Having given the radius and the axis of a cylinder of revolution, to pass a plane tangent to the cylinder and parallel to a given straight line without finding the projections of the cylinder.

13. A cone of revolution is given by its axis and the angle which the elements make with the axis. Pass a plane tangent to the cone through a given point.

14. A cone of revolution is given by its axis and the angle which the elements make with the axis. Pass a plane tangent to the cone and parallel to a given straight line.

15. A circle which lies in a plane parallel with H is the base of a cone. The vertex of the cone is in G. L. Through some other point on G. L., pass a plane tangent to the cone. Show the angles which the plane makes with the planes of projection.

16. Find a common normal to two cylinders of revolution.

17. To two given cylinders, pass tangent planes which are parallel to each other.

18. To a given cone and cylinder, pass tangent planes which are parallel to each other.

19. Two cones with their bases on H have the same altitude. Pass a plane tangent to each so that the planes will be parallel.

20. Through a given point, pass a plane tangent to two given spheres.

21. Pass a plane tangent to two given spheres and parallel to a given straight line.

22. Pass a plane tangent to three given spheres.

23. Through a given point, pass a plane which is equidistant from three given spheres.

24. Through a given straight line, pass a plane which is equidistant from two given spheres.

25. Pass a plane which is equidistant from four given spheres.

26. Find a common tangent plane to a sphere and a cylinder of revolution.

27. Find a common tangent plane to a sphere and a cone of revolution.

28. Having given the traces of a plane which contains an element of a hyperboloid of revolution of one sheet, to determine its point of tangency with the surface.

CHAPTER IV

PLANE SECTIONS AND DEVELOPMENTS OF SURFACES

117. To find the intersection of a surface by a plane. Pass a system of auxiliary planes through the surface. Each of these planes will cut a line from the surface and a straight line from the given plane which will intersect in a point of the required line of intersection. The auxiliary planes should be taken so as to cut the simplest lines from the surface, straight lines or circles if possible.

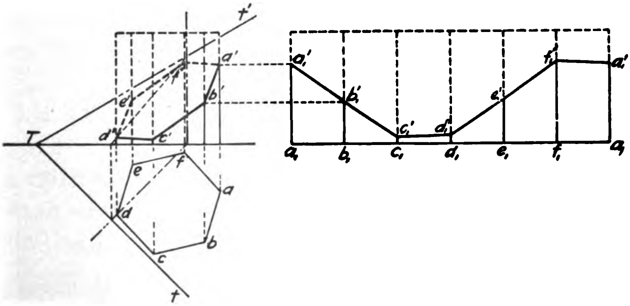
If the intersection is a broken line, find the points where it changes direction. If it is a curved line, the points where it is tangent to the extreme elements of the surface and enough other points to determine the curve should be found.

By the **development** of a surface is meant a plane area equal to that of the surface, and of such a form that it can be rolled up again into the shape of the surface. The development is formed by rolling the surface upon a tangent plane until each element of the surface in turn comes into the plane.

118. To find the intersection of a right hexagonal prism by a plane and to develop the surface of the prism.

Let the plane and prism be given as in Fig. 65.

Fig. 65



Plane section and development of right prism

Analysis. Pass the system of auxiliary planes through the lateral edges of the prism. These planes will cut lines from the given plane which will cross the edges of the prism in points of the required line of intersection.

Following the above analysis, the intersection ABCDEF, Fig. 65, is found.

To develop the surface of the prism showing its line of intersection with the plane T. If one of the lateral faces of the prism be placed against a plane and then the prism be revolved about an edge of this face until the next face comes into this plane, and the process be repeated until each face in turn is brought into the plane, the part of the plane covered by the different faces of the prism will be the development of the prism. In Fig. 65, $a_1b_1=ab$, $b_1c_1=bc$, etc. To get the line of intersection on the development, lay off a_1a_1' equal to the distance of a' above the ground line, b_1b_1' equal to b' from the ground line, etc., and join these points by straight lines.

Note. All the problems in plane sections of prisms and pyramids can be multiplied by changing the number of lateral faces of the surface.

119. To find the intersection of a right circular cylinder by a plane and to develop the surface of the cylinder.

Analysis. Intersect the cylinder and the plane by a system of auxiliary planes containing the rectilinear elements of the cylinder. These planes cut elements from the cylinder and straight lines from the given plane. The intersections of these straight lines and elements are points on the required curve of intersection.

Let the construction be made in accordance with the above analysis.

To draw a tangent to the curve of intersection at any point. Select the point at which the tangent is to be drawn and then pass a plane tangent to the cylinder at this point (Art. 89). The intersection of this plane with the cutting plane is the required tangent (Art. 81).

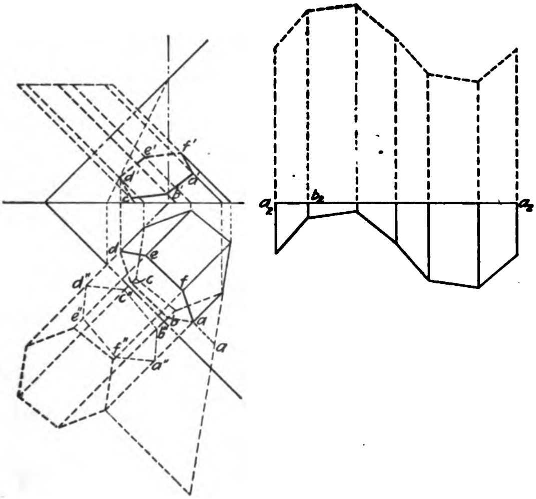
To develop the surface of the part of the cylinder included

between the given plane and H. Let a regular prism be inscribed in the cylinder. Now let the number of faces of the prism increase, always keeping the new prisms inscribed in the cylinder, until a chord which is a side of the base of the prism is equal approximately to the arc which it subtends. Then develop this new prism by the method of Art. 118; joining the upper ends of the elements by a smooth curve in the place of straight lines. This gives the required development of the cylinder.

120. To find the intersection of an oblique hexagonal prism by a plane perpendicular to the lateral edges, and to develop the lateral surface of the prism.

Let the prism and plane be given as in Fig. 66.

Fig. 66



Plane section and development of oblique prism

By passing the auxiliary planes through the lateral edges of the prism and perpendicular to H, the intersection ABCDEF is found.

To develop the lateral surface of the prism. Since the section plane is perpendicular to the edges of the prism, it is evident that the line of intersection will be a straight line in the development. The true size of the section is $a'' b'' c''$, etc. This is found by revolving it into the horizontal plane about t T as an axis. In the development lay off $a_2 b_2 = a'' b''$, $b_2 c_2 = b'' c''$, etc., along the straight line $a_2 a_2$. On a line perpendicular to $a_2 a_2$ at the point a_2, lay off below the line the distance from A along the edge to the lower base of the prism and along the same line lay off above $a_2 a_2$ the distance from A to the upper base of the prism. The length of this edge is found by revolving it into the horizontal plane. In a similar manner, lay off at the point b_2 the distance along the edge from B to the lower base below the line $a_2 a_2$, and the distance from B to the upper base above $a_2 a_2$. In this way all the points in both bases of the prism are located in the development. Straight lines joining in order the points in the development of the lower base and also in the same way those of the upper base complete the development of the prism.

121. To find the intersection of an oblique cylinder by a plane and to develop the surface of the cylinder.

Let the plane be taken perpendicular to the elements of the cylinder. This is not necessary in order to get the intersection but it is necessary for the development.

Analysis. Cut the cylinder and plane by a system of auxiliary planes parallel with the elements of the cylinder and perpendicular to H. The elements cut from the cylinder will intersect the lines cut from the plane in points of the required curve of intersection.

Let the construction be made in accordance with the above analysis.

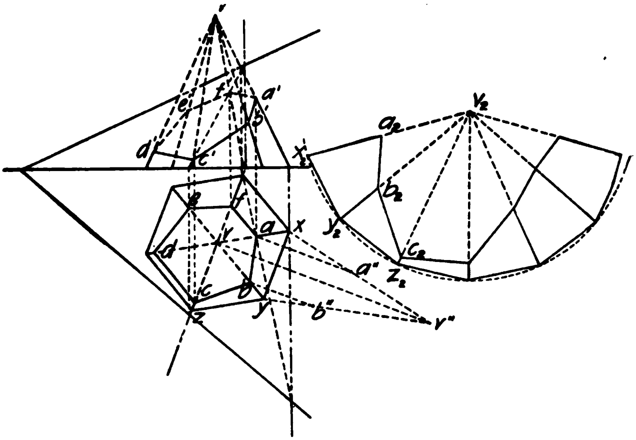
To draw a tangent to the curve of intersection at a given point. Select the point at which the tangent is to be drawn. The intersection of a tangent plane to the surface at this point with the plane of the section is the required tangent.

To develop the suface of the cylinder. Let an oblique prism be inscribed in the cylinder. Increase the number of faces of the prism, always keeping the new prism inscribed in the cylinder, until a chord which is a side of the base of the prism is equal approximately to the arc which it subtends. Then develop the prism as in the preceding article, joining the ends of the elements with smooth curves in the place of straight lines. This gives the required development. The plane section will develop into a straight line while the base of the cylinder will become a curve.

122. To find the intersection of a regular hexagonal pyramid by a plane and to develop the lateral surface of the pyramid.

Let the plane and pyramid be given as in Fig. 67.

Fig. 67



Plane section and development of right pyramid

By passing the auxiliary planes through the lateral edges of the pyramid, the intersection ABCDEF is found.

To develop the lateral surface of the pyramid. If the face VXY of the pyramid be placed against a plane and the pyramid rolled over about VY as an axis until the face VYZ comes into the plane, and then the pyramid rolled again about VZ as an axis until the next face comes into the plane, etc., the part of the plane thus covered is the development of the pyramid. To construct the development, strike an arc with the true length of VX as a radius. Apply the distance XY six times as a chord to this arc and join these points with each other and also with the center of the arc. This completes the development of the pyramid.

The line of intersection of the pyramid with the plane can be laid off on the development by laying off on each edge in succession the true length of VA, VB, etc. and joining these points by straight lines.

123. To find the intersection of a right circular cone by a plane and to develop the surface of the cone.

Analysis. Cut the cone and plane by a system of planes through the elements of the cone and perpendicular to H. The elements cut from the cone will intersect the lines cut from the plane in points of the required curve of intersection.

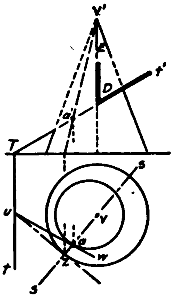
Let the construction be made in accordance with the above analysis.

To draw a tangent to the curve of intersection at a given point. Select the point at which the tangent is to be drawn. The intersection of the tangent plane to the cone at this point with the plane of the section is the required tangent.

To develop the surface of the cone. Let a regular pyramid be inscribed in the cone. Then increase the number of faces of the pyramid, always keeping the new pyramid inscribed in the cone, until a chord which is a side of the base of the pyramid is equal approximately to the arc which it subtends. Then develop the pyramid as in the preceding article, joining the ends of the elements with a smooth curve in the place of straight lines. This gives the required development.

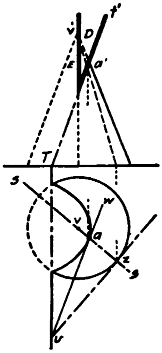
The line of intersection of the cone by the plane is laid off on the development in the same way that it is on the development of the pyramid, Art. 122. The points should be joined with a curve in the place of straight lines.

Fig. 68



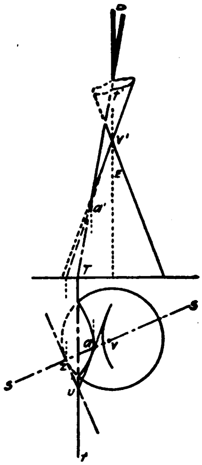
Ellipse

Fig. 69



Parabola

Fig. 70



Hyperbola

Conic sections

124. Plane sections of a right circular cone.

Let the cone be given as in Figs. 68, 69 and 70, E being the angle which the elements of the cone make with the axis.

Circle. When the plane of the section is perpendicular to the axis of the cone, the section is a circle.

Ellipse. When the angle D, Fig. 68, which the plane of the section makes with the axis of the cone, is less than 90° and greater than E, the section is an ellipse.

Parabola. When the angle D, Fig. 69, which the plane of the section makes with the axis of the cone, is equal to E, the section is a parabola.

Hyperbola. When the angle D, Fig. 70, which the plane of the section makes with the axis of the cone, is less than E, the section is the hyperbola.

When the plane passes through the vertex, it will cut straight line elements from the cone providing it cuts the cone in any point besides the vertex.

125. To find the intersection of an oblique hexagonal pyramid by a plane and to develop the lateral surface of the pyramid.

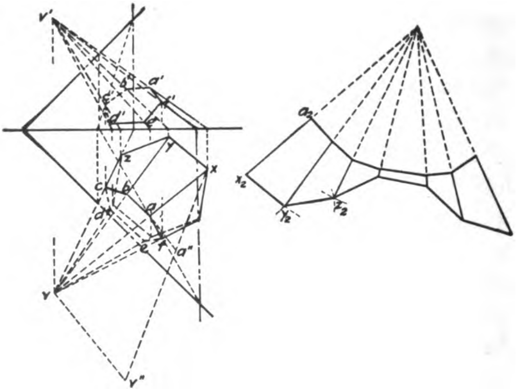
Let the plane and pyramid be given as in Fig. 71.

By passing the auxiliary planes through the lateral edges of the pyramid, the intersection ABCDEF is found.

To develop the lateral surface of the pyramid. If the face VXY of the pyramid be placed against a plane and the pyramid rolled over about VY as an axis until the face VYZ comes into the plane and then the pyramid rolled again about VZ as an axis until the next face comes into the plane, etc., the part of the plane thus covered is the development of the pyramid. Hence the development is found by constructing a series of six triangles, having given the lengths of the sides of each triangle. The first triangle is formed of the sides VX, VY and XY; the second of the sides VY, VZ and YZ, etc.

The line of intersection of the pyramid with the plane T can be laid off on the development by laying off on each edge in succession the true lengths of VA, VB, etc., and joining these points by straight lines.

Fig. 71



Plane section and development of oblique pyramid

126. To find the intersection of an oblique cone by a plane and to develop the surface of the cone.

Analysis. Cut the cone and plane by a system of planes through the vertex of the cone and perpendicular to H. The elements cut from the cone will intersect the lines cut from the plane in points of the required curve of intersection.

Let the construction be made in accordance with the above analysis.

To draw a tangent to the curve of intersection at a given point. The intersection of a tangent plane to the cone at this point with the plane of the section is the required tangent.

To develop the surface of the cone. Let an oblique pyramid be inscribed in the cone. Then increase the number of faces of the pyramid, always keeping the new pyramid inscribed in the cone, until a chord which is a side of the base of the pyramid is equal approximately to the arc which it subtends, and then develop the pyramid as in Art. 125; joining the ends of the elements with a smooth curve in the place of straight lines. This gives the required development.

The line of intersection of the cone by the plane is laid off on the development in the same way that it is on the development of the pyramid, Art. 125. The points should be joined with a curve in the place of straight lines.

127. To find the intersection of a warped surface by a plane.

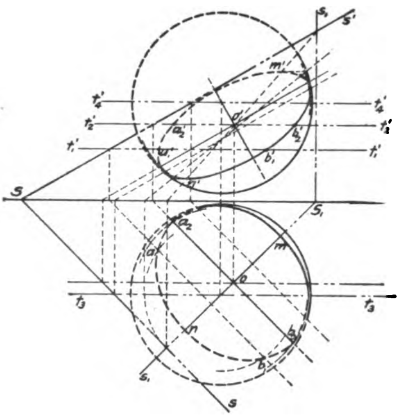
Analysis. A system of planes cutting straight line elements from the surface will cut lines from the section plane. The intersections of these elements and lines will be points of the required curve of intersection. In case the surface has a plane directer, the auxiliary planes should be parallel with it.

128. To find the intersection of any surface of revolution by a plane.

Let the plane and sphere be given as in Fig. 72.

Analysis. Cut the surface and plane by a system of auxiliary planes to perpendicular the axis of the surface. The auxiliary

Fig. 72



Plane section of a surface of revolution

planes will cut circles from the surface which will intersect the lines cut from the given plane in points of the required curve of intersection.

Construction. Since the method is the same for all surfaces of revolution, let the construction be made for the sphere as given in Fig. 72. The planes whose vertical traces are $t_1't_1'$, t_2' t_2', etc., are parallel with H and cut lines from the plane S which are parallel to the horizontal trace sS, and intersect the circles cut from the sphere in the points A_1, B_1; A_2, B_2; etc., points of the required curve of intersection. By continuing in this manner enough points can be found to determine both projections of the curve of intersection.

To determine particular points of the curve of intersection. To find the points where the curve of intersection touches the great circle which represents the vertical projection of the sphere, use an auxiliary plane T_3 through the center of the sphere and parallel to V. To find where the curve touches the great circle which represents the horizontal projection of the sphere use an auxiliary plane T_4 through the center of the sphere, and parallel to H. To find the highest and lowest points on the curve of intersection, use a plane S_1 through the center of the sphere and perpendicular to the horizontal trace of the plane S. Revolve the plane S_1 about an axis perpendicular to H through the center of the sphere until it is parallel to V. The revolved position of the line of intersection of the planes S and S_1 is shown in the figure, the point on the axis remaining fixed during the revolution. This line will intersect the vertical projection of the sphere in the highest and lowest points revolved.

By the counter revolution of the plane S_1, the required points M and N are found. To find the major axis of the ellipse in each view: first find the center O of the circle cut out and then pass planes through this center parallel to H or V. The figure shows the construction for these axes. The minor axis is perpendicular to the major axis at its middle point. The method of finding the major axis does not apply to all surfaces of revolution.

To draw a tangent to the curve of intersection at a given point. Select the point at which the tangent is to be drawn. The intersection of the tangent plane to the sphere at this point with the plane S is the required tangent.

PROBLEMS

1. Given a square prism with axis perpendicular to H. Side of base $2\frac{1}{4}''$. Cut a parallelogram from the prism having one side $3''$ long and one angle 60°.

2. Given a square prism with axis perpendicular to H. Side of base $2\frac{1}{4}''$. Cut a parallelogram from the prism having one diagonal $6\frac{1}{2}''$ long and one angle 60°.

3. Assume any four points which are not in the same plane, and draw through them four parallel lines in such a way that the plane sections of the prism, which has the four lines for edges, will be parallelograms.

4. Cut a parallelogram having one side $2''$ long from the pyramid O—ABCD. O $(2\frac{1}{2}, \frac{5}{8}, 5\frac{1}{2})$, A $(-4\frac{1}{4}, 4\frac{3}{4}, 0)$, B $(2, 5\frac{3}{4}, 0)$, C $(2\frac{7}{8}, 3\frac{1}{8}, 0)$, D $(-\frac{3}{8}, \frac{5}{8}, 0)$.

5. Given a point on the edge of any tetrahedron which rests on H. Find the shortest path to be followed on the faces in order to go around the tetrahedron and return to the same point. The path is to cross the three faces but is not to cross the base.

6. Two intersecting planes cut the ground line at the points T and S. Find the horizontal and vertical projections of the shortest path on the planes from T to S.

7. Pass a plane so as to cut a regular hexagon from a $3''$ cube.

8. Is the shortest path around a cone, following the surface, starting at a point and returning to the same point, a plane curve?

9. Find the intersection of a hyperbolic paraboloid by a plane.

10. Find the intersection of a conoid by an oblique plane.

11. Find the intersection of a cylindroid by an oblique plane.

12. Find the intersection of a helicoid by a plane.

13. Find the intersection of an oblate ellipsoid of revolution by a plane.

14. Find the intersection of the hyperboloid of revolution of one sheet by a plane.

15. From a right circular cone with base on H, cut an ellipse and show the true size of the section.

16. From a right circular cone with base on H, cut a parabola and show the true size of the section.

17. From a right circular cone with base on H, cut an hyperbola and show the true size of the section.

18. An oblique cone has a circle on H for a base. Cut a circle of given radius from this cone by a plane which is not parallel to the base.

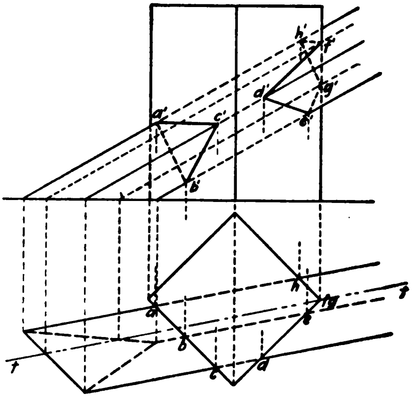
CHAPTER V

INTERSECTIONS OF SURFACES

129. To find the line of intersection of any two surfaces. In general, pass a system of auxiliary planes through the surfaces. The planes will cut lines from each surface and the intersections of these lines will be points on the required line of intersection. The auxiliary planes should be passed in such a way as to cut the simplest lines possible from the given surfaces.

Sometimes the intersection can be more easily found by using a system of auxiliary spheres in place of the system of planes. For example, if the axes of two surfaces of revolution intersect, a system of spheres having their centers at the intersection of the axes and radii of different lengths, will cut circles from each of the original surfaces. The intersections of these circles are points on the required line of intersection.

Fig. 73



Intersection of prisms

130. To find the line of intersection of the surfaces of two prisms when one of them is perpendicular to the horizontal plane.

Let the prisms be given as in Fig. 73.

Analysis. Pass a system of auxiliary planes parallel to the lateral edges of both prisms. These planes will cut straight lines from each prism which will intersect in points of the required line of intersection.

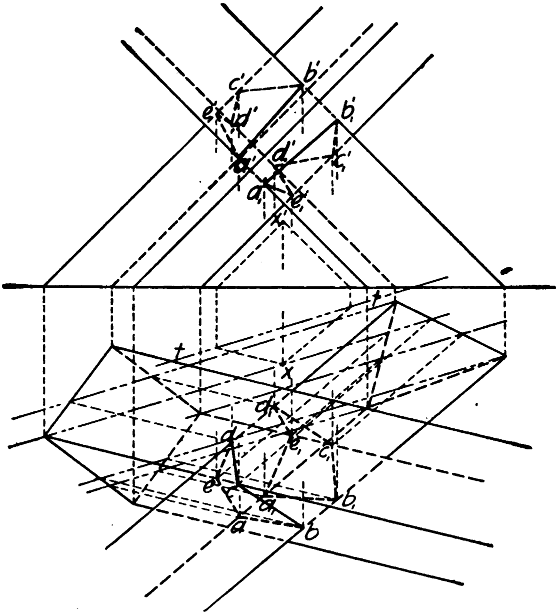
Construction. Since the line of intersection will be a broken line, it is only necessary to find where the edges of each prism pierce the faces of the other prism. The auxiliary planes in this case are perpendicular to H. The construction is shown in Fig. 73.

The visible part of the line of intersection must be on a visible face of each prism.

131. To find the line of intersection of the surfaces of two oblique prisms.

Let the prisms be given as in Fig. 74.

Fig. 74



Intersection of oblique prisms

Analysis. Pass a system of auxiliary planes parallel to the lateral edges of both prisms. These planes will cut straight lines from each prism which will intersect in points of the required line of intersection.

Construction. Through any point as X, draw lines parallel to the edges of each prism. tt, which is the horizontal trace of the plane containing these lines, gives the direction of the horizontal traces of the auxiliary planes. The line of intersection is a broken line which only changes direction at the points where the edges of one prism pierce the faces of the other. By using only the planes which contain edges of one or the other of the prisms and joining in order the points thus found, the broken line of intersection as shown in Fig. 74 is found.

132. To find the line of intersection of two oblique cylinders.

Analysis. Pass a system of auxiliary planes parallel to the elements of both cylinders. These planes will cut straight line elements from each cylinder which will intersect in points of the required curve of intersection.

Construction. The construction is the same as that for the intersection of two prisms, Art. 131. Enough points should be found on the curve of intersection so that the curve can be drawn in accurately with the curve-ruler. It is always desirable to locate the points where the curve of intersection touches the extreme elements of the cylinders in both horizontal and vertical projections. To do this, draw the traces of the auxiliary plane through the point where the element in question pierces the corresponding plane of projection. The intersection of the elements thus cut out will give the required point.

To draw a tangent to the curve of intersection at any point. Select the point at which the tangent is to be drawn and pass a plane tangent to each cylinder at this point. The intersection of these planes is the required tangent to the curve.

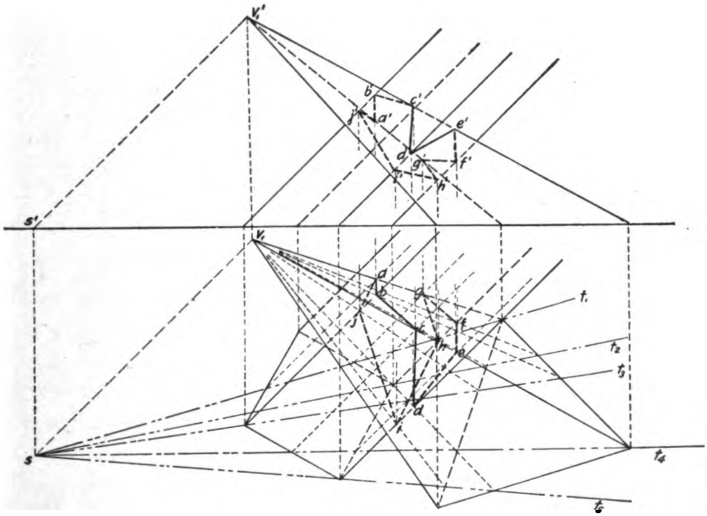
To determine the visible part of the curve of intersection. A point on the curve of intersection is visible in the horizontal view if it lies on an element of each cylinder which comes from the visible part of the base of the cylinder in that view. In a similar way, a point to be visible in the vertical view must lie on an element of each cylinder which comes from the visible part of the base of the cylinder in that view.

133. To find the line of intersection of the surfaces of a prism and a pyramid.

Let the prism and pyramid be given as in Fig. 75.

Analysis. Pass a system of auxiliary planes through the vertex of the pyramid and parallel with the lateral edges of the prism. These planes will cut straight lines from each surface which will intersect in points of the required line of intersection. A line through the vertex of the pyramid and parallel to the lateral edges of the prism will lie in all of these auxiliary planes and will therefore pierce H in a point common to all their horizontal traces.

Fig. 75



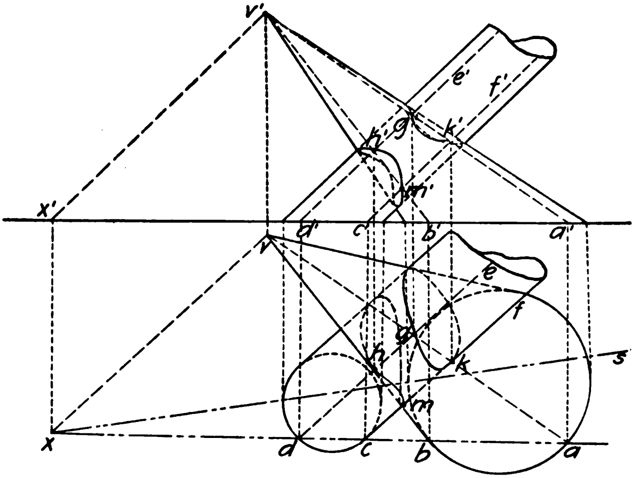
Intersection of prism and pyramid

Construction. Through V_1, draw V_1S parallel with the lateral edges of the prism. It pierces H at s. Through s, draw st_1, st_2, st_3, etc., horizontal traces of the auxiliary planes. The intersections of the lines which these planes cut from the pyramid and prism are points on the required line of intersection. By using only the planes which contain edges of the pyramid or the prism and joining in order, with straight lines, the points thus found, the line of intersection as shown in Fig. 75 is found.

134. To find the line of intersection of a cone and cylinder.

Let the surfaces be given as in Fig. 76.

Fig. 76



Intersection of cone and cylinder

Analysis. Pass a system of auxiliary planes through the vertex of the cone and parallel with the elements of the cylinder. These planes will cut from each surface straight line elements which will intersect in points of the required line of intersection. A line through the vertex of the cone and parallel with the elements of the cylinder will lie in all of the auxiliary planes.

Construction. The line VX, through the vertex of the cone and parallel with the elements of the cylinder, lies in all the auxiliary planes and pierces H at x, a point common to all of the horizontal traces. Through x, draw the traces xa, xs, etc., cutting the bases of both surfaces. From the points where these traces cross the bases draw elements of the surfaces. These elements will intersect in points of the required line of intersection. In this manner the curve of intersection as shown in Fig. 76 is found.

The visible part of the curve is determined and the tangent is drawn in the same manner as in Art. 132.

135. To determine in advance the nature of the curves in which cylinders and cones intersect. In the system of auxiliary planes used in Fig. 76, there are two which are tangent to the cone. The horizontal trace of one runs from x tangent to the base of the cone on the lower side and the horizontal trace of the other is tangent to the base of the cone on the upper side. Both of these tangent planes miss the cylinder, the cylinder being between them. It is evident then that the cone extends beyond the cylinder on both sides; the cylinder running into the cone on one side and out on the other side, forming two distinct curves of intersection, as is shown in the figure. If one of these tangent planes had cut the base of the cylinder while the other did not, there would have been one continuous curve of intersection. If both of these planes had cut the base of the cylinder, there would again have been two distinct curves of intersection, where the cone runs into the cylinder on one side and where it runs out on the other. In the case where the planes are tangent to both cone and cylinder, the two curves of intersection just touch each other.

The same method may be used to determine the nature of the curve of intersection of two cylinders or of two cones.

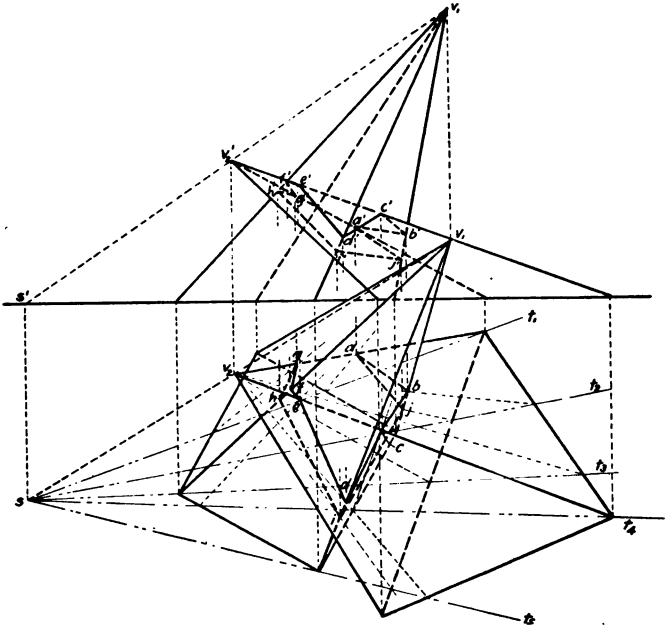
136. To find the line of intersection of the surfaces of two pyramids.

Let the pyramids be given as in Fig. 77.

Analysis. Pass a system of auxiliary planes through the vertices of both pyramids. These planes will cut straight lines from each pyramid which will intersect in points of the required line of intersection. A line joining the vertices of the pyramids will lie in all of the auxiliary planes and will pierce H in a point common to all their horizontal traces.

Construction. The line through the vertices V_1 and V_2 pierces H at s. Through s, draw the horizontal traces st_1, st_2, st_3, etc. The intersections of the lines which these planes cut from the pyramids are points on the required line of intersection. By using only the planes which contain edges of the pyramids and joining in order the points thus found by straight lines, the line of intersection as shown in Fig. 77 is found.

Fig. 77



Intersection of pyramids

137. To find the line of intersection of two cones.

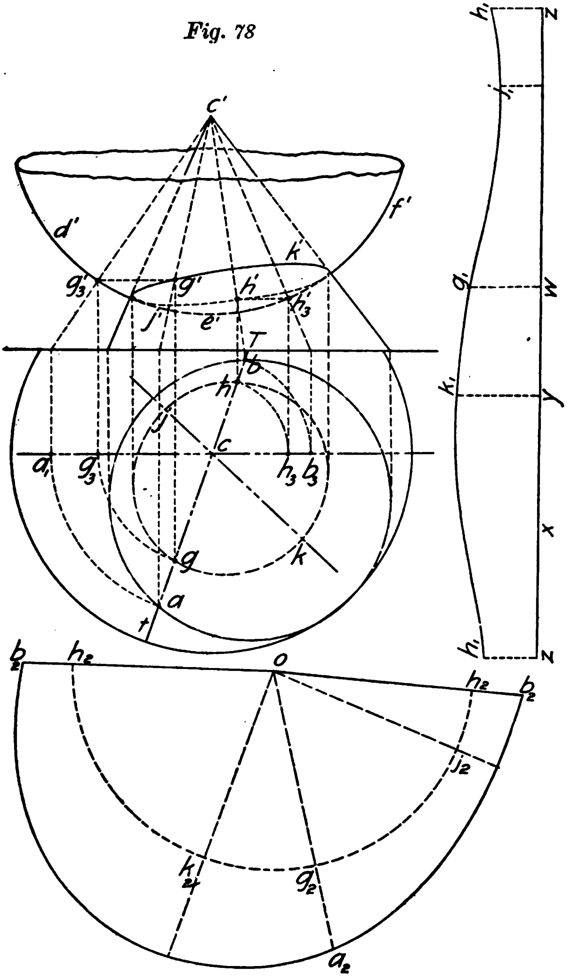
Analysis. Use the same method as was used for finding the intersection of two pyramids.

Let the construction be made for finding the curve of intersection. Also determine which part of the curve is visible and draw a tangent to the curve by methods previously given.

138. To find the line of intersection of a cone and a sphere when the vertex of the cone is at the center of the sphere.

Let the surfaces be given as in Fig. 78.

Fig. 78



Intersection of oblique cone and sphere. Development of oblique cone

Analysis. Intersect the surfaces by a system of auxiliary planes through the vertex of the cone and perpendicular to the horizontal plane. Each of the auxiliary planes will cut one or more elements from the cone (Art. 91) and a great circle from the sphere which will intersect these elements in points of the required curve of intersection.

The vertical projections of the circles cut from the sphere are ellipses. In the place of drawing these ellipses, it is more convenient to revolve the auxiliary planes about an axis perpendicular to H through the vertex of the cone until they become parallel with V. In this position, the elements cut from the cone and the circle cut from the sphere are shown in their true size. After finding the intersection of the elements and circle the planes can be revolved back to the original position and the required points located.

Construction. tT is the horizontal trace of a plane through the vertex C perpendicular to H. This plane will cut the elements CA and CB from the cone and a great circle from the sphere. When the plane T is revolved about a vertical axis through C until it is parallel to V, the elements CA and CB will take the positions CA_1 and CB_3 and the great circle the position DEF. In the revolved position, the elements intersect the great circle in the points G_3 and H_3, two points of the required curve of intersection. In the counter revolution, these points move in horizontal circles to the positions G and H. In the same manner any plane through the vertex perpendicular to H will determine other points of the intersection.

To draw a tangent to the curve at any point. The required tangent will be the line of intersection of two planes, one tangent to the cone and the other tangent to the sphere at the given point.

139. To develop an oblique cone.

Let the cone be given as in the preceding problem, Fig. 78.

Analysis. As every point of the curve of intersection of the cone and sphere in the preceding problem is the same distance from the vertex of the cone, this curve will develop into the arc of a circle having a radius equal to the radius of the sphere. The distance between any two elements of the cone in the development, measured along this circle, is equal to the length of the curve of intersection between those elements on the surface of the cone. Any curve on the surface of the cone, as the base, may be found by laying off in development the true lengths of the rectilinear elements from the vertex of the cone to the points where they cut the curve.

Construction. The true length of the curve of intersection is found by developing its horizontal projecting cylinder. The base of this cylinder, which is the horizontal projection of the curve, will develop into the straight line zz. The elements zh_1, yk_1, wg_1, etc., in development will be at right angles to this line, and will be shown in their true lengths and their true distances from each other measured on the base, khjg, of the cylinder. Their true lengths are shown in the vertical view. Hence $h_1k_1h_1$ is the true length of the curve of intersection. With o as a center and the radius of the sphere as a radius, describe the indefinite arc, $h_2k_2h_2$. On this arc lay off h_2k_2, k_2g_2, g_2j_2, j_2h_2, equal respectively to h_1k_1, k_1g_1, g_1j_1, j_1h_1, and draw the elements in development from the vertex o through these points. On these elements lay off their true lengths $c'b'_3$, $c'a'_1$, etc., locating the points b_2, a_2, etc., on the developed base. It is necessary to take more elements than are shown in the cut in order to get an accurate development.

This method of developing the oblique cone is not as desirable, practically, as that given in article 126.

140. To find the line of intersection of a cylinder and a sphere.

Analysis. Pass a system of auxiliary planes through the elements of the cylinder and perpendicular to one of the planes of projection. These planes will cut elements from the cylinder and circles from the sphere which will intersect in points of the required curve of intersection. To find these points, revolve the auxiliary planes into the plane of projection to which they are perpendicular. In this position the elements of the cylinder and the circles of the sphere are shown in true size and the points of intersection easily found. By revolving these planes back to their original positions, the projections of the required points are found.

Let the construction be made in accordance with the above analysis.

PROBLEMS

1. Find the points in which a straight line pierces a given oblique pyramid, prism, cone or cylinder. Also find where it pierces a sphere or an ellipsoid.

2. The base of a polyhedron is determined by the following four points: A $(-2\frac{3}{4}, 2\frac{3}{8}, 0)$, B $(-1\frac{1}{2}, \frac{3}{8}, 0)$, C $(-2\frac{1}{4}, 3\frac{7}{8}, 0)$, D $(\frac{3}{8}, 2\frac{3}{8}, 0)$. The V projections of the edges of the polyhedron make 60° and the H projections—30° with G. L. The upper base is parallel to H and $5''$ above it. Find the intersection of this polyhedron with the pyramid O—EFG. O$(-3, \frac{3}{8}, 5\frac{1}{4})$, E$(-1\frac{1}{8}, 4\frac{3}{4}, 0)$, F $(2\frac{1}{8}, 1\frac{1}{2}, 0)$, G $(3\frac{1}{4}, 4\frac{1}{2}, 0)$.

3. Find the intersection of a cube with a square prism when the diagonal of the cube coincides with the axis of the prism. Edge of cube $2\frac{1}{2}''$; side of base of prism $2\frac{1}{4}''$; altitude $5''$. Axis of prism perpendicular to H, the plane of one face making 15° with V. The H projection of one top edge of the cube is perpendicular to a face of the prism.

Develop the surface of the cube showing the line of intersection.

4. Same as problem 3 except that the prism has an equilateral triangle with the side $2\frac{1}{2}''$ for a base.

5. Same as problem 3 except that the diagonal of the cube is $\frac{1}{2}''$ from the axis of the prism.

6. Find the line of intersection of a square prism and a right square pyramid. Prism $2'' \times 2'' \times 4\frac{1}{2}''$, with edges parallel to H and 30° with V. The plane of one face makes 30° with H and the lowest edge is $\frac{3}{4}''$ above H. Base of pyramid $2\frac{1}{2}''$ square resting on H. Altitude $5''$. Side of base makes 60° with V. Axis of pyramid perpendicular to H. Axes of prism and pyramid $\frac{1}{8}''$ apart.

Note: This problem can be multiplied by changing the distance between the axes of the prism and pyramid.

7. Same as problem 6 except that the base of the prism is an equilateral triangle with side $2''$.

8. Find the intersection of an oblique cone and a right circular cylinder. The bases of both surfaces are circles on H.

9. Find the intersection of a right circular cone with a sphere. The base of the cone is on H and the center of the sphere not on the axis of the cone.

10. Find the intersection of a right circular cylinder with a right circular cone. The axes of both surfaces are pependicular to H but do not coincide.

11. Find the intersection of a sphere with a right circular cylinder. The axis of the cylinder is perpendicular to H but does not pass through the center of the sphere.

12. Find the intersection of a regular square pyramid with a sphere. The base of the pyramid is inscribed in a great circle of the sphere.

13. Find the intersection of a right circular cone with a right circular cylinder. Base of cone on H and axis of cylinder parallel to the ground line.

Draw a tangent to the curve of intersection.

14. Find the intersection of a triangular prism with a sphere.

15. Find the intersection of a cone with a triangular prism. The base of the prism is inscribed in the base of the cone.

16. Find the intersection of an oblique cylinder with an oblique cone. The base of the cylinder is on H and the base of the cone is on V.

17. Find the intersection of two oblique cylinders; the base of one is on H and the base of the other is on V.

18. Find the intersection of two oblique cones. The bases are circles on H, one within the other but not concentric. The vertices do not coincide.

19. The axis of a right circular cylinder is parallel to the ground line. Find the intersection of this cylinder with a right rectangular prism having its axis perpendicular to V.

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RM. 340 - WENDT LIBRARY

DEMCO

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